A New Approach to Model Verification, Falsification and Selection

by

Andrew J. Buck
Department of Economics
Temple University

George M. Lady
Department of Economics
Temple University

Department of Economics
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and

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Abstract

This paper shows that a qualitative analysis can always be used in evaluating a model's validity both in general and compared to other hypothesized models. The analysis relates the sign patterns and possibly other information of hypothesized structural arrays to the sign pattern of the estimated reduced form. It is demonstrated that such an analysis can <u>always</u> potentially falsify the hypothesized structural sign patterns or support an analysis of the relative likelihoods of alternative structural hypotheses, if neither are falsified. It is also noted that a partially specified structural hypothesis can be sometimes falsified by estimating as few as one reduced form equation. Additionally, zero restrictions in the structure can themselves be falsified; and, when so, current practice proposes estimated structural arrays that are impossible. It is further shown how the information content of the hypothesized structural sign patterns can be measured using Shannon's (1948) concept of entropy. In general, the lower the hypothesized structural sign pattern's entropy, the more *a priori* information it proposes about the sign pattern of the estimated reduced form. As an hypothesized structural sign pattern has a lower entropy, it is more subject to type 1 error and less subject to type 2 error.

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Department of Economics

Temple University

Philadelphia, PA

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I. Introduction. There are some issues to work through in demonstrating the accuracy of the statement that economics is a science. In general, such a demonstration would be comprised of showing that economics organizes its subject matter in some fashion that limits the configurations that may be taken on by the data. Organizing the subject matter in a fashion that limits the permissible realizations of the data may be termed "theory." If the limits on the data required by the theory are not found, then the theory has been falsified (Popper(1934)). The potential for falsification provided by propositions about the economy is termed by Samuelson (1947) as the source of economics' "meaningful theorems."

For example, an allocation-of-time model might produce the result that more schooling results in greater lifetime earnings, *ceteris paribus*. It is the all-other-things-equal clause that produces the wrinkle. Angrist and Krueger (1991) suggested that the date of birth sorts children into groups with more or less education for a given chronological age, and therefore we can observe the effect of schooling on earnings independent of ability. Butcher and Case (1995) argue that it is gender of first-born-sibling that can be used to identify the value of schooling. These two papers might be thought of as (non-)nested alternatives.[1] Barron, Ewing and Waddell (2000) consider the effect of high school athletics participation on lifetime earnings differences. Since they do not account for quarter of birth or gender of first-born-sibling their theory imposes zero restrictions on the theories of the first two papers. Not surprisingly all three papers report statistically significant returns to educational activity, be it sports participation or years of schooling. All three of these papers use instrumental variables estimation. None of them have asked to what extent the alternative structural models restrict the outcomes in the first stage of estimation. The implicit first stage reduced form is not fraught with the statistical difficulties inherent in reporting structural results common to all three papers and would therefore be the proper seat for model falsification. Determination of the permissible signs for the reduced form coefficients, given the structural model, is known as a "qualitative analysis." To state the issue

[1] Rosenzweig and Wolpin (2000) show that both the Angrist-Krueger and the Butcher-Case models produce biased results, but in opposite directions. The implication is that both models are wrong, which begs the question of why both studies show statistically significant results.

succinctly: It is quite possible to fit a 'model' or 'theory' to the data and find statistically significant results even though the model could not possibly have generated the observed data.[2]

This paper shows that a qualitative analysis can always be used in evaluating a model's validity both in general and compared to other hypothesized models. Issues that can be approached via qualitative analysis are identified and manners of assessing them are illustrated. To do this, algorithmic approaches are used to process a structure's sign pattern information in order to reach conclusions about a model's scientific content, its potential for falsification (and associated type 1 error) and issues at stake in its acceptance (and associated type 2 error). It is demonstrated that such an analysis can always potentially falsify the hypothesized structural sign patterns or support an analysis of the relative likelihoods of alternative structural hypotheses, if neither are falsified. It is also noted that a partially specified structural hypothesis can be sometimes falsified by estimating as few as one reduced form equation. Additionally, zero restrictions in the structure can themselves be falsified; and, when so, current practice proposes estimated structural arrays that are impossible. It is further shown how the information content of the hypothesized structural sign patterns can be measured using Shannon's (1948) concept of entropy. In general, the lower the hypothesized structural sign pattern's entropy, the more *a priori* information it proposes about the sign pattern of the estimated reduced form. As an hypothesized structural sign pattern has a lower entropy, it is more subject to type 1 error and less subject to type 2 error.

Background. Following Samuelson, we propose that economic "theory" about some feature of the economy is expressed by an enumeration of n-many endogenous variables Y to be explained by m-many exogenous variables Z. Further, relationships among these variables are expressed by n-many relationships,

$$f^{i}(Y, Z) = 0, \; i = 1, 2, \ldots, n. \qquad (1)$$

[2] At the risk of belaboring the point, consider a model car. The model car is not an exact replica of a real car, but we could ask what part of the real car is correctly described by the model. What we show in this paper is that empirical implementation of the model car may lead us to believe that we have correctly described some part of the real car, even though the model could not possibly have generated the observed data.

A propos of the returns to schooling papers cited above, the issue of whether or not the system (1) presents a refutable hypothesis is studied by the method of comparative statics, as specified by a linear system of differentials associated with disturbing a solution (which we assume always exists) to the system (1),

$$\sum_{j=1}^{n} \frac{\partial f^i}{\partial y_j} dy_j + \sum_{k=1}^{m} \frac{\partial f^i}{\partial z_k} dz_k = 0, i = 1, 2, \ldots, n. \quad (2)$$

For the econometrician to bring the system (2) to the data, it is often assumed that the relationships in (2) are (at least locally) linear. Adding unobserved disturbance terms expressed by the n-vector U, (2) is given by the system below,

$$\beta Y = \gamma Z, + \delta U, \qquad (3)$$

where β, γ and δ are appropriately dimensioned matrices. (3) is usually called the *structural form.* Bringing (3) to the data requires manipulation. To avoid a variety of statistical issues the system to be estimated is expressed as,

$$Y = \pi Z + \psi U, \qquad (4)$$

where $\pi = \beta^{-1}\gamma$. (4) is usually called the *reduced form,* from which the unknowns in (3) are subsequently derived. Given this, the issue of falsification becomes that of the limitations on the entries of the estimated $\pi = \beta^{-1}\gamma$, denoted by $\hat{\pi}$, as derived from the theoretical proposals about the entries of {β, γ}.

When Samuelson, *op. cit.,* considered the issue in exactly these terms, he first noted that the theory might specify the directions of influence among the endogenous and exogenous variables, i.e., the sign patterns of β and γ. Given this, due to a "qualitative analysis," it might be possible to go through the algebra of calculating $\pi = \beta^{-1}\gamma$ and finding that certain entries of π must have particular signs, based entirely upon the sign patterns of β and γ, independent of the magnitudes of their entries. If so, then when $\hat{\pi}$ is estimated, if the required signs do not show up, the theory has been falsified. On reflection, Samuelson proposed that as a practical matter, a qualitative analysis was extremely unlikely to be successful. To work, all of the potentially millions of terms in the expansions of β's determinant and cofactors in computing β^{-1} would all have to have the same sign, independent of magnitudes. Samuelson viewed this as too unlikely to be taken seriously. Accordingly, he proposed other sources for limitations on the entries, i.e., signs, of π, e.g., that β is a stable matrix or is derived from the second order conditions to a (perhaps constrained) optimization problem.

Nevertheless, a literature on the conditions such that a successful qualitative analysis could be conducted was developed (well summarized in Hale, *et al,* (1999)). For example, a liberal sprinkling of zeros among β's entries can drastically reduce the number of nonzero terms in the expansions of its determinant and cofactors. Still, even so, the conditions for a successful qualitative analysis were found to be severe and are rarely satisfied.[3] Accordingly, there is virtually no tradition of assessing a model's qualitative properties as might be at issue in falsifying entries in sgn π. And actually, Samuelson's other criteria are rarely explored as well. It is tempting to worry that the degree to which economic models are "scientific;" that is, falsifiable based on the data, is really not well in-hand.

The purpose of this paper is to demonstrate that the above view of the potential usefulness of qualitative analysis is miscast: starting with Samuelson's pessimism through the development of the extremely restrictive conditions such that entries of π can be shown to have particular signs, based upon a qualitative analysis. Instead, we show that a qualitative analysis can always be successfully conducted and provides important insight into the "scientific" content of an economic model and can always identify outcomes for sgn π that would falsify hypothesized sign patterns for the structural arrays.

In particular, it will be shown that:

- a specification of the sign patterns of the structural arrays always limits the possible sign patterns that can be taken on by the estimated reduced form;

[3] Hale and Lady (1995) is an interesting exception.

- the falsifiability of hypothesized sign patterns for the structural arrays is independent of identifiability;

- zero restrictions specified for the structural arrays can be falsified independent of the sign pattern of the nonzero entries. When this happens, multi-stage least squares will always be in error when providing estimates of the structural arrays, i.e., the sign patterns of the estimated structural arrays with zero restrictions imposed are impossible, given the sign pattern of the estimated reduced form;

- a partially specified structural hypothesis sometimes can be falsified by a small number of estimated reduced form equations, even as few as one;

- the conditional probability that an hypothesized structural sign pattern is actually true when not falsified can be estimated using Bayes' formula;

- the relative likelihood of competing structural hypotheses can be assessed, given the sign pattern of the estimated reduced form, based upon their qualitative characteristics; and,

- hypothesized structural arrays can be assessed for their information content using Shannon's (1948) measure of entropy.

In the next section, a method of qualitative falsification is presented, including using a measure of entropy to measure the information content of a structural hypothesis. In the section following that, issues in "verifying" a structural hypothesis that is not falsified are discussed. The last section provides a summary. An appendix provides additional detail about our method of analysis.

II. Qualitative Falsification. Lady and Buck (2011) and Buck and Lady (2012) showed that the falsification of a qualitative specification of a model does not require that any individual signs of β^{-1} (or more generally sgn π) be signable based upon sgn β (and more generally, also sgn γ). Instead, patterns of signs in sgn π may not be possible, even though no individual entry is signable. In this section this principle is generalized compared to this earlier work. In the next section the new issue of verification is presented.

An easy example to use for explication is β: 2 x 2 without zero entries. For this and all of the examples below, it is assumed that β is irreducible and not singular. There are sixteen possible sign patterns that can be hypothesized for β. Of these, eight have an entry with a sign the opposite of the other three. For these cases, the determinant of β is signable, independent of the magnitudes of its entries. Since the adjoint of β is signable, for these cases there is only one possible sign pattern for β^{-1}. For the other eight cases, there are only two possible sign patterns for β^{-1}; however, no entry of sgn β^{-1} is the same for these two possibilities, since they are the negative of each other. Accordingly, for any of these eight cases, an hypothesized sgn β would be falsified by sgn π (assuming for the moment that $\gamma = I$) if it took on one of the fourteen sign patterns that are not possible for the hypothesized sgn β, even though no individual entry of sgn β^{-1} is signable.

For larger arrays (and/or for $\gamma \neq I$), although the same principles apply, the algorithmic burden of revealing them quickly becomes more difficult. To cope with this, we developed a Monte Carlo algorithm for investigating the possible sign patterns of the reduced form based upon a qualitative specification of the structural form.[4] In brief, sgn β and sgn γ (if need be) are randomly given quantitative values for their entries consistent with the hypothesized sign pattern, $\pi = \beta^{-1}\gamma$ is computed, and the sign pattern outcome of this computation is saved. The sampling can be done millions of times. For sufficiently small systems, the set of all possible sign patterns that the reduced form might take on, given the hypothesized sign pattern of the structural arrays, can be tabulated with a vanishingly small potential for error as the sample size increases. A central point of the analysis is that this set of allowed sign patterns always limits the sign patterns the reduced form can take on. Given this, there are <u>always</u> sign patterns that the reduced form array might take on that are impossible, given the hypothesized structural sign patterns, and if so, the given structure is thus, falsified.

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[4] This algorithmic approach and its use are briefly described in the Appendix,

Consider as the next example the case of hypothesizing that sgn β is 3 x 3 and all positive and γ = I.[5] In principle, there are 512 such 3 x 3 sign patterns that sgn β^{-1} can take (barring zeros as we always do for the examples we provide, although this is not necessary). Of these, only 102 were found by the Monte Carlo in tens of millions of iterations. Given this, it appears that 410 sign patterns for sgn β^{-1} are impossible for the hypothesized sgn β. The analytic basis for some sign patterns being impossible can be quickly developed. Let the qualitative inverse of sgn β be defined as:

Definition (qualitative inverse). Sgn π is a qualitative inverse of sgn β if and only if there exist magnitudes for the entries of β, consistent with the given sgn β, such that sgn β^{-1} = sgn π.

Given this, it is immediate that the proposed sign patterns must be such that magnitudes can be assigned so that:

Sgn π is a qualitative inverse of sgn β only if it is possible that i) βπ = I; and, ii) πβ = I.

This possibility can be algorithmically investigated directly. For β 3 x 3 and all positive, of the 512 possible 3 x 3 sign patterns that sgn π might take on, only 216 sign patterns satisfy i) or ii) individually; and, there are only 102 sign patterns that satisfy both. These are what the Monte Carlo found. Since i) and ii) are clearly necessary, for the case here their satisfaction also turned out to be sufficient for sgn π to be a qualitative inverse for the hypothesized sgn β. As we will see below, satisfaction of both of these conditions may not always be sufficient. From the standpoint of falsification for β all positive and of any dimension, it is immediate that a proposed, i.e., estimated, sgn π cannot satisfy both i) and ii) unless it has a positive and negative entry in each row and column. More generally, sgn π cannot be a qualitative inverse of a proposed sgn β unless, for i) above the i^{th} row of β and the j^{th} column of π share a pair of common nonzero entries of the same sign for i = j and both this and another pair of common

[5] This is not a trivial example even though the system is exactly identified. In standard practice the estimates of the structural coefficients can be solved from the reduced form using indirect least squares. In spite of being exactly identified the proposed model can still be wrong, produce a set of first stage results that would by our methodology falsify the proposed model, yet produce estimates of some of the proposed model coefficients that were of the proposed sign and be statistically significant.

nonzero entries of the opposite sign for i ≠ j; and, also for the i^{th} row of π and the j^{th} column of β for ii) above.

For the special case that we are considering, why not simply set $\hat{\beta} = \hat{\pi}^{-1}$ and compare sgn $\hat{\beta}$ with the hypothesized sgn β? If any signs are different, then the hypothesized sgn β has been falsified and there is no need to worry about the derivations immediately above. This could indeed be done for the exactly identified case, which given our simplifying assumptions requires that β have no zero restrictions. But, even for this very special case, if sgn $\hat{\beta} \neq$ sgn β, this outcome might not be taken as falsification. It might be that sgn $\hat{\pi}$ is a qualitative inverse of both sgn $\hat{\beta}$ and the hypothesized sgn β. Given this, further analysis, not to speak of empirical work, might be viewed as necessary. This situation is discussed in the next section. Generally, for n ≠ m and/or γ ≠ I setting $\hat{\beta} = \hat{\pi}^{-1}$ is not possible and/or appropriate. For these more general cases, if (3) is exactly identified, then unique arrays $\hat{\beta}$ *and* $\hat{\gamma}$ can be found based upon $\hat{\pi}$; nevertheless, as pointed out above, this may not lead to falsification, at least not without further analysis.

For the special case at issue here, if zero restrictions are imposed upon sgn β, then (3) is over identified and the zero restrictions will be imposed upon the derivation of $\hat{\beta}$. Under these circumstances, there is more than one way to recover $\hat{\beta}$ from $\hat{\pi}$. For these, sgn $\hat{\beta}$ may not be the same. Even more serious, the zero restrictions themselves can be falsified. If this is the case for the sgn $\hat{\pi}$ that was found, whatever method is used to recover $\hat{\beta}$ results in a mistake if these zero restrictions are imposed; namely, the estimated array $\hat{\beta}$ is impossible, given sgn $\hat{\pi}$; i.e., sgn $\hat{\pi}$ is not a qualitative inverse of sgn $\hat{\beta}$.

As an easy to follow example, let the hypothesized β be given by,

$$\text{sgn } \beta = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ 0 & ? & ? \end{bmatrix}.$$

For this array, there is the single zero restriction $\beta_{31} = 0$. The remaining entries marked "?" are nonzero, but can be either positive or negative. For this case, assume that,

$$\text{sgn}\,\hat{\pi} = \begin{bmatrix} ? & ? & ? \\ + & - & ? \\ + & + & ? \end{bmatrix}.$$[6]

This result cannot be the qualitative inverse for any array for which $\beta_{31} = 0$, since the (1,3) cofactor is signable, independent of magnitudes. Of the 512 possible sign patterns that sgn $\hat{\pi}$ might take on, thirty-two of them have the above 2 x 2 sign pattern corresponding to the (1,3) cofactor. As discussed above, altogether there are eight such signable sign patterns for this cofactor. As a result, fully 256 of the possible 512 sign patterns that sgn $\hat{\pi}$ might take on falsify the given zero restriction, independent of the signs of the other entries of sgn β or sgn $\hat{\pi}$. If any of these show up for sgn $\hat{\pi}$, then however $\hat{\beta}$ is derived, it is a mistake, i.e., impossible, if the zero restriction is imposed.[7]

To dramatize the potential for the zero restrictions themselves to falsify a structural hypothesis, and potentially lead to an error when deriving $\hat{\beta}$ from $\hat{\pi}$, consider as an example an inference structure for β (assumed to be irreducible) that contains only one cycle of inference. Consider this case for n = 5,

$$\text{sgn}\,\beta = \begin{bmatrix} ? & 0 & 0 & 0 & ? \\ ? & ? & 0 & 0 & 0 \\ 0 & ? & ? & 0 & 0 \\ 0 & 0 & ? & ? & 0 \\ 0 & 0 & 0 & ? & ? \end{bmatrix}.$$

[6] Strictly, sgn a = 1, -1, 0 as a > 0, a < 0, a = 0. We will use the signs themselves as a felicitous convention.
[7] For our example, the condition $\gamma = I$ is also imposed. Accordingly, it is this condition plus $\beta_{31} = 0$ that is being falsified, not just the zero restriction on β.

As before, the entries marked "?" are nonzero, but may be of any sign. The number of possible qualitative inverses for this array can be derived directly. In particular, the expansion of each cofactor of β includes only one term, since there is only one path of inference between any pair of (endogenous) variables. There are ten nonzeros which can take on $2^{10} = 1024$ distinct sign patterns. Accordingly, adjoint β can only take on 1024 distinct sign patterns. These in turn can be divided into two groups for which the patterns in one group are the negative of the patterns in the other group. As it happens, the determinant is signed for half of the 1024 cases, but this doesn't make any difference. Given the possible sign patterns of the adjoint, there are only 1024 possible sign patterns for sgn β^{-1}, regardless of how the determinant's (nonzero) sign plays out. Yet, for a 5 x 5 array, ba**r**ring zeros, there are 33,554,432 possible sign patterns. For this case, of these, 33,553,408 are impossible, based upon the zero restrictions on β alone (plus $\gamma = I$). For this inference structure, the proportion of qualitative inverses among the possible sign patterns decreases significantly as n increases, since there are 2^{nn} possible sign patterns for which only 2^{2n} are qualitative inverses. It is worth it to emphasize the points just made. If the zero restrictions of the structural hypothesis are falsified by the sign pattern of the estimated reduced form, then the estimated structural arrays with these zero restrictions imposed are impossible, given the sign pattern of the estimated reduced form. Since current practice does not include conducting a qualitative analysis, this leaves the prospect of the outcome of many estimations as unknowingly being utterly impossible.

If two or more entries of the adjoint of β are signable, which could well be the case if the nonzeros in β are sparse, then a criterion in addition to i) and ii) above for the falsification of β based upon sgn π is in-hand; namely, the entries of sgn π that correspond to the signable entries of the adjoint must always have the same or opposite signs as appropriate, or the hypothesized β is falsified. This circumstance can be robustly detected by the Monte Carlo and algorithmically applied directly. To organize this idea formally, let $B = [B_{ij}]$ be the adjoint of β.

Definition (adjoint-consistent): If sgn B_{ij} = sgn B_{uv} (resp. sgn B_{ij} = - sgn B_{uv}) independent of magnitudes, then sgn π is *adjoint-consistent* with sgn β if and only if sgn π_{ij} = sgn π_{uv} (resp. sgn π_{ij} = - sgn π_{uv}).

Thus, a third criterion would be,

iii) Sgn $\hat{\pi}$ is a qualitative inverse of sgn β only if sgn $\hat{\pi}$ is adjoint consistent with sgn β.

Once two or more signable entries in sgn B are identified, satisfaction of iii) can be algorithmically investigated directly. Taken together, i), ii), and iii) are necessary. As shown in the example below, they may not be sufficient. The ultimate requirement for qualitative inverses of sgn β would be the existence of a solution to a system of inequalities, as given below.

For a given sgn $\hat{\boldsymbol{\pi}}$, consider the system(s) of inequalities as written out symbolically as the expansions of β's cofactors and determinant,

$$\text{sgn B} = \text{sgn } \hat{\boldsymbol{\pi}}, \text{ and det } \beta > 0\text{; and/or, sgn B} = -\text{ sgn } \hat{\boldsymbol{\pi}}, \text{ and det } \beta < 0. \quad (5)$$

It is immediate that a given sgn $\hat{\boldsymbol{\pi}}$ is a qualitative inverse of a proposed sgn β if and only if at least one of the systems (5) has a solution. Even if i), ii), and iii) are satisfied, a given sgn $\hat{\pi}$ may not be such that one or both of the systems (5) has a solution. The necessary and sufficient algorithmic method for assessing if (5) is satisfied, if it exists, is beyond the scope of this paper. It was the resolution of this issue (among others) for which the Monte Carlo was developed as an heuristic. For a given sgn $\hat{\pi}$, if the hypothesized sgn β is quantitatively sampled repeatedly and β^{-1} computed, and it is repeatedly found that, given this, the given sgn $\hat{\pi} \neq$ sgn β^{-1}, then the inference is that the given sgn $\hat{\pi}$ is not a qualitative inverse of the hypothesized sgn β. The chances of incorrectly falsifying the hypothesized structure are made ever smaller by increasing the number of quantitative samples taken, but the chances are never absolutely zero. For this reason, once falsification is strongly suspected based upon the Monte Carlo, a direct investigation of (5) can be attempted, hoping to analytically confirm what the Monte Carlo suggests. Examples of this approach are given in the appendix of Buck and Lady (2012).

As an example of applying all of the criteria presented above, consider the case (with γ = I),

$$\text{sgn}\,\beta = \begin{bmatrix} - & + & + & 0 \\ + & - & + & 0 \\ 0 & + & - & + \\ 0 & + & + & - \end{bmatrix}.$$

This sign pattern, negative main diagonal entries and non-negative off diagonal entries, is an example of a Metzler (1945) matrix, a form corresponding to the excess demand functions for multimarket equilibria for which all commodities are (weakly, if zeros are allowed) gross substitutes. The four zero restrictions are imposed here for the purpose of demonstrating all of the falsification criteria discussed above.

Application of the above criteria revealed the following: Of the 65,536 possible 4 x 4 sign patterns for sgn β^{-1}, only 4,096 are possible, given the zero restrictions, independent of the signs of the nonzeros. To reiterate this point yet again, if the sign pattern of the estimated reduced form had any of the other 61,440 sign patterns, then any sign pattern proposed for sgn $\hat{\beta}$ with these zero restrictions imposed would be a mistake, i.e., the estimated reduced form is not a qualitative inverse of the proposed $\hat{\beta}$. Direct analysis revealed that only 625 of the possible sign patterns for sgn β^{-1} satisfied i) above, 400 satisfied ii), and only 28 satisfied both. Further analysis revealed that entries of β adjoint were signable, as given below,

$$\text{Adjoint}\,\beta = \begin{bmatrix} * & * & a & a \\ b & b & a & a \\ a & a & \text{c} & c \\ a & a & * & * \end{bmatrix}.$$

In the above, entries marked "a," "b," and "c" always have the same sign. Taking this into account for the 28 sign patterns that satisfied both i) and ii), only 20 satisfied iii), i.e., presented the pattern of equal signs given above. Of these, the Monte Carlo only found 18. Since the outcome of i), ii), and iii) can be inspected directly, the findings are clearly necessary. Of 20 sign patterns that satisfied all of these necessary conditions, two were nevertheless not solutions of

(5). For these two, further analysis would be appropriate to confirm that the finding falsified the hypothesized structure, i.e., to uncover the contradiction that prevents the satisfaction of (5).

A feature of the above analysis was the complete specification of the qualitative structure of (3) and (4). It should be noted that falsification might be the case if only parts of the structure are qualitatively specified and less than all of the reduced form equations are estimated. For example, if the sign pattern of only one column of β is hypothesized, then if only one row of π is estimated, the (partial) structural hypothesis has been falsified if the signs of the entries of that row are the opposite from the corresponding nonzeros in the hypothesized column of β, since the criterion ii) above would not be satisfied. Other examples of falsifying partially specified structural models can be readily proposed.

The simplifying assumptions for the analysis so far, n = m and $\gamma = I$, were made to permit the principles of the analysis to be readily revealed. Relaxing these simplifications increases the analytic burden of the analysis; nevertheless, a qualitative specification of the structural form will inevitably impose restrictions on β^{-1} that always result in these restrictions being translated to the reduced form in the more general cases.

For the structural form exactly- or over-identified, estimates $\hat{\beta}$ *and* $\hat{\gamma}$ can be derived, although they may not be unique and may be otherwise problematical. Accordingly, the sign patterns of such estimates may not be viewed as decisive from the standpoint of falsification; unlike qualitative falsification, which is decisive. For the under-identified structural form, such estimates cannot be made; nevertheless, qualitative falsification can proceed as before.

Consider the structural hypothesis:

$$\operatorname{sgn}\beta = \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix} \; \textit{and} \; \operatorname{sgn}\gamma = \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}.$$

For this under-identified structural form, estimates for $\hat{\beta}$ *and* $\hat{\gamma}$ cannot be made. Nevertheless, qualitative analysis shows that of the 512 3 x 3 reduced form sign patterns, only 343 are possible for this sign pattern for the structural form. The structural hypothesis would be falsified if any of the other 169 sign patterns were estimated for the reduced form.

Qualitative Falsification and Type 1 Error. As noted above, a structural hypothesis may be rejected because the sign pattern of the estimated reduced form is not found among those admissible reduced forms generated by the Monte Carlo sampling algorithm that we have employed. Given this, further investigation is called for in order to reveal the inconsistency as related to a solution to (5). If this is found, i.e., if the sign pattern of the estimated reduced form is demonstrably impossible, given the hypothesized structural sign pattern, the result may be due to errors in the data, and to reject the structural hypothesis would constitute a type 1 error. The issue of this subsection is to assess the propensity for a structural hypothesis to be incorrectly rejected.

As an example (to be used in the next section as well), let the structural sign patterns given below be termed "system #1:"

$$\operatorname{sgn}\beta = \begin{bmatrix} + & + & + & 0 \\ + & + & + & + \\ + & + & + & + \\ 0 & 0 & + & + \end{bmatrix} \textit{ and } \operatorname{sgn}\gamma = \begin{bmatrix} + & 0 & 0 & 0 \\ 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \end{bmatrix}.$$

Of the 65536 4 x 4 sign patterns that the estimated reduced form might take on, simulation results report that only 535 are possible, which constitute less than 1% of the total number. The Monte Carlo simulation draws the absolute values of the nonzeros in the above arrays from a uniform distribution over the open interval]0,10[. This is sufficient to potentially find all possible reduced form sign patterns. But suppose the distributional rule for populating β and γ is not otherwise supported by data or analysis, which can make a difference as pointed out below. An indicative method to portray the potential for type 1 one error is to draw 'error terms' from a normal distribution with mean zero and add these to each nonzero entry used to populate the sample arrays, and then compute $\pi = \beta^{-1}\gamma$. If the sign pattern found is not one of the (535 for system #1) possible sign patterns, then the associated falsification is a type 1 error due to the error terms introduced into the quantitative sampling. For this example, this can only happen if the error term that was drawn puts a negative value in one of the array entries when added to the sampled quantitative value that was supposed to be positive. Such a "wrong" sign is intended to represent errors in the data. As the number of possible sign patterns permitted by the structural sign patterns is smaller, the potential for type 1 error due to errors in the data is larger. For system #1 above, with the standard deviation of the 'error term' = .2, there were 4.68% of 500,000 quantitative samplings of the system #1 sign patterns falsified. This result is dependent in part on the distributional assumption employed in the sampling. This influence of the sampling method on this outcome can be further investigated by the Monte Carlo simulation, but it is beyond our scope to do so.

For the purpose of comparison, a second structural system was formulated, basically by moving three zero restrictions from γ to β and otherwise changing two signs in β. Given this, system #2 is specified as:

$$\operatorname{sgn}\beta = \begin{bmatrix} + & - & 0 & 0 \\ 0 & + & + & 0 \\ + & 0 & + & + \\ - & 0 & + & + \end{bmatrix} \textit{ and } \operatorname{sgn}\gamma = \begin{bmatrix} + & - & 0 & 0 \\ - & + & 0 & 0 \\ 0 & - & + & 0 \\ 0 & 0 & 0 & + \end{bmatrix}.$$

The Monte Carlo reports that for system #2, that there are only 65 possible sign patterns for the reduced form. From the above discussion, since there are fewer possible reduced form sign patterns (or better yet, a smaller proportion of the total), the potential for type 1 error would be larger for system #2 compared to system #1. This result is reflected in the results tabulated in Table 1 below.

Table 1: Proportion of Type 1 Error

Standard Deviation of the Error Distribution	**Proportion Falsified (500,000 samples)**	
	System #1	**System #2**
.2	.0468	.0672
.4	.0976	.1395

.6	.1420	.1964
.8	.1847	.2604
1	.2366	.3256

Type 1 Error and Entropy. The result of the last subsection can be generalized somewhat. Consider that for the sign pattern of an n x m reduced form array, barring zeroes, that there are nm bits of information (say "1" for "+" and "0" for "-"), one bit for each entry. Given an hypothesized structural sign pattern, the possible reduced form sign patterns are limited to $Q < 2^{nm}$; and for these, given the algebra of computing $\pi = \beta^{-1}\gamma$ and the distributional rules for assigning values to the nonzeros of $\{\beta, \gamma\}$ each possible reduced form sign pattern has a particular likelihood of occurrence. Let F_i be the frequency of occurrence of the ith possible reduced form sign pattern (see below for how "i" is computed). . Given this, the entropy of the distribution (Shannon (1948), see also Cover and Thomas (1991)) is given by:

$$Entropy(\text{sgn}\,\beta, \text{sgn}\,\gamma) = -\sum_{i \in Q} F_i \log(F_i), \quad (6)$$

where $\log(F_i)$ is to the base 2. For example, for one simulation of system #1 above with 3,000,000 samples taken, the entropy of the resulting distribution of (the 535 possible) reduced form sign patterns was 8.18. The unit of this measure is bits. It measures the information achieved by estimating the reduced form. The maximum entropy is 16 for the 4 x 4 sign pattern (barring zeros) and would be computed if all of the 65,536 4 x 4 reduced form sign patterns were possible and equally likely. The minimum entropy is zero if only one sign pattern for the reduced form is possible. Our algorithm for sampling the structural sign pattern is designed to find the possible reduced form sign patterns, but has no econometric justification for the distributional (uniform) rule used in the sampling. Nevertheless, we will use the frequency distributions found in the analysis below. If a default entropy is computed which "simply" treats all of the possible reduced form sign patterns as equally likely, then doing this for system #1 causes the entropy measure to increase a small amount to 9.06. The key item driving the measure is the number of possible sign patterns, as opposed to the frequency distribution of the possibilities.

As the entropy measure is high, the structural hypothesis provides less information about the expected outcome of the reduced form's estimated sign pattern, and conversely. To revise the measure to be larger as the information provided by the structural hypothesis is greater, the following amended measure is proposed,

$$INFO\%(\operatorname{sgn}\beta, \operatorname{sgn}\gamma)) = 100(1 - \frac{Entropy(\operatorname{sgn}\beta, \operatorname{sgn}\gamma)}{nm}). \quad (7)$$

For system #1 this comes out to 48.94% and for system #2 (which is more limiting) it comes out to 66.4%. Accordingly, system #2 provides more *a priori* information about the expected outcome of the estimated reduced form's sign pattern (only 65 are possible compared to 535 for system #1) and thus is more susceptible to type 1 error compared to system #1.

III. Qualitative Verification. Although contingent upon the propensity for type 1 error, qualitative falsification is otherwise entirely decisive. Given the hypothesized sign pattern of the structural form, either the estimated reduced form sign pattern is possible, or it is not. If not, subject to error, the structural hypothesis has been falsified.

On-the-other-hand, if the sign pattern of a structural hypothesis is not falsified, this does not establish that the structural hypothesis is "true." Indeed, as Samuelson (*op. cit.)* pointed out, if degenerate hypotheses are allowed, there are a virtually unlimited number of hypothesized structural forms that could be consistent with, i.e., not falsified by, a particular outcome for the estimated reduced form. In fact, science does not establish "the" true hypothesized system. Instead, a system may be used that is persistently consistent with the data, perhaps chosen by, say, the principle of Occam's Razor; but always, such a system awaits its replacement by another system that matches its explanatory power to the present point, but provides further explanation of data that the initial system cannot explain, e.g., such as features of the orbit of Mercury around the Sun that were explained by general relativity, but were not explained by Newtonian physics.[8]As a result, what we will term a model's "verification" is an open-ended, in some sense never-ending, exercise of continuous testing, evaluation, and potential comparison with competing hypotheses. Open-ended or not, the point of this section is to present a number of procedures due to which a qualitative analysis can help evaluate the validity of an hypothesis and compare it to competitors.

[8] See Isaacson (2007), Chapter 9.

The analysis that we develop here will be conditional upon the lists of endogenous and exogenous variables. It will always be assumed that β is irreducible and that γ is otherwise configured such that the n x m reduced form has no logical zeros; and further, we will assume that there are no zeros otherwise in the estimated reduced form. In this frame of reference, a type 2 error is the acceptance of an hypothesized structural form due to its not being falsified, even though the hypothesized structure is not correct. The propensity for a type 2 error can be assessed in general or in comparison to one or more competing hypotheses.

The latter issue might be addressed by assessing the likelihood that an hypothesized structural sign pattern might fail to be falsified by accident, i.e., if the sign pattern of the estimated reduced form were in some sense random and was found to be a member of the permissible set of reduced forms. From this perspective, it is intuitive that system #2 is less likely to be randomly accepted compared to system #1, since the number of possible outcomes for the sign pattern of the reduced form is substantially less for system #2. There is nothing wrong with this insight, but the analysis can be pushed further, specifically in terms of estimating the conditional probability for each system's validity, given the outcome of the estimated reduced form sign pattern, p(system #1 or #2|sgn $\hat{\pi}$). Given this, the likelihood of type 2 error is one minus this conditional probability.

For the two systems proposed for illustration here, of the possible reduced form sign patterns for each, twenty-eight are common to both of them. We will develop our illustration in terms of these twenty-eight sign patterns. To start, the Monte Carlo sampling (3,000,000 repetitions for each system) provides an immediate estimate of the conditional probability of a reduced form sign pattern for each system as provided by the frequency with which that sign pattern appeared. These results are given in Table 2 below for the two hypothesized systems. The reduced form sign pattern for the first row of Table 2 is:

$$\text{sgn}\ \hat{\pi}(8914) = \begin{bmatrix} - & - & + & - \\ - & - & + & - \\ + & + & - & + \\ - & - & + & - \end{bmatrix}.$$

The base 2 index for this sign pattern is 0010001011010010 ("0" for "-" and "1" for "+") which equals 8914 in base 10. The base 10 indices in the other rows are computed in the same way and are found in the first column of the table. In the second column of the table, the sign pattern of the reduced form at issue is written out row by row.

Not surprisingly, the frequencies for system #2 are uniformly larger than system #1 since significantly fewer sign patterns are possible for this system, although exceptions are possible. The estimates in columns 4 and 6 can be used in computing the desired conditional probability using Bayes' formula:

$$p(system\#1\ or\#2\,|\,\mathrm{sgn}\,\hat{\pi}) = \frac{p(\mathrm{sgn}\,\hat{\pi}\,|\,system\#1\ or\ \#2)\,p(system\#1\ or\ \#2)}{p(\mathrm{sgn}\,\hat{\pi})} \quad (8).$$

To use this formula, in addition to the estimates of the conditional probabilities provided in Table 2, estimates of the prior probabilities for each system and the reduced form sign pattern must also be made. Presumably, in a particular empirical context, there would be particular reasons for assigning these values. Here, for the sake of illustration, we will proceed as follows:

Consider that for the thirty-two entries in β and γ, the two hypothesized systems agree on the signs for twenty-three of them. Given this, for all combinations of the nine disputed signs, there are 512 (= 2^9) possible structural sign patterns. Assume that all of these are equally likely; hence, the prior probability for each such system = 1/512 = .00195, including systems #1 and #2. These 512 structural sign patterns might be termed the "universe" from which the specific structural sign patterns were selected, i.e., as what the theory dictates and (evidently) leaves undecided.[9] This universe can be sampled and the frequency found for each of the twenty-eight reduced form patterns enumerated in Table 2. This was done for a sample of 300,000,000 and the results are reported in the appendix, as well as the work up for computing the conditional probabilities for

[9] It might be noted that this hypothesized "universe" can itself be falsified. For example, an all positive reduced form is impossible for any of the 512 structural sign patterns.

each system for each reduced form sign pattern as given in equation (8). The estimates found for these conditional probabilities are given in Table 3 below.

Table 2: Frequency of Reduced Form Sign Patterns in Sample

π Base 10 Index	Sign Pattern of the Reduced Form Row by Row	Frequency in Sample of 3,000,000			
		System #1		System #2	
		Count	Frequency: p(π |sys#1)	Count	Frequency: p(π |sys#2)
8914	- - + - - - + - + + - + - - + -	3101	0.00103	84455	0.02815
8915	- - + - - - + - + + - + - - + +	1661	0.00055	70168	0.02339
9878	- - + - - + + - + - - + - + + -	2448	0.00082	17047	0.00568
9879	- - + - - + + - + - - + - + + +	3642	0.00121	28049	0.00935
9938	- - + - - + + - + + - + - - + -	2329	0.00078	14731	0.00491
9939	- - + - - + + - + + - + - - + +	4576	0.00153	84842	0.02828
21925	- + - + - + - + + - + - - + - +	1933	0.00064	122556	0.04850
26262	- + + - - + + - + - - + - + + -	10809	0.00360	51052	0.01702
26263	- + + - - + + - + - - + - + ++	6590	0.00220	42724	0.01424
38189	+ - - + - + - + - - + - + + - +	1703	0.00057	18889	0.0063
38249	+ - - + - + - + - + + - + - - +	879	0.00029	36755	0.01225
39273	+ - - + + - - + - + + - + - - +	7996	0.00267	77355	0.02578
40237	+ - - + + + - + - - + - + + - +	2442	0.00081	8728	0.00291
40297	+ - - + + + - + - + + - + - - +	4291	0.00143	16501	0.0055
41562	+ - + - - - + - - + - + + - + -	560	0.00019	44339	0.01478
41563	+ - + - - - + - - + - + + - + +	1061	0.00035	99335	0.03311
41682	+ - + - - - + - + + - + - - + -	1137	0.00038	43087	0.01436
41683	+ - + - - - + - + + - + - - + +	1713	0.00057	28838	0.00961
42586	+ - + - - + + - - + - + + - + -	2884	0.00096	20251	0.00675
42587	+ - + - - + + - - + - + + - + +	7693	0.00256	165480	0.05516
42646	+ - + - - + + - + - - + - + + -	5229	0.00174	182054	0.06068
42647	+ - + - - + + - + - - + - + + +	10518	0.00351	98504	0.03283
42706	+ - + - - + + - + + - + - - + -	7833	0.00261	48281	0.01609
42707	+ - + - - + + - + + - + - - + +	26405	0.00880	75320	00.02511
43610	+ - + - + - + - - + - + + - + -	2029	0.00068	70759	0.02359
43611	+ - + - + - + - - + - + + - + +	1690	0.00056	76690	0.02556
54573	+ + - + - + - + - - + - + + - +	1280	0.00043	56623	0.01887
56621	+ + - + + + - + - - + - + + - +	7170	0.00239	55421	0.01847

Table 3: Estimated Conditional Probabilities for System #1 and System #2

sgn $\hat{\pi}$ **Base 10 Index**	**P(System #1|sgn $\hat{\pi}$)**	**P(System #2|sgn $\hat{\pi}$)**
8914	.00044	.01214
8915	.00014	.00597
9878	.00188	.01303
9879	.00085	.00656
9938	.00108	.00679
9939	.00020	.00372
21925	.00010	.00725
26262	.00291	.01377
26263	.00052	.00334
38189	.00278	.03071
38249	.00018	.00773
39273	.00221	.02130
40237	.00255	.00915
40297	.00216	.00831
41562	.00016	.01259
41563	.00013	.01225
41682	.00028	.01057
41683	.00009	.00152
42586	.00082	.00580
42587	.00067	.01454
42646	.00065	.02276
42647	.00044	.00413
42706	.00206	.01270
42707	.00074	.00211
43610	.00043	.01508
43611	.00023	.01043
54573	.00015	.00676
56621	.00057	.00439

Inspection of Table 3 reveals that for each of the twenty-eight reduced form sign patterns, system #2 is more likely than system #1. Since the prior probabilities of each system and the reduced form sign pattern are the same for each system, this inequality will be the same as the conditional probabilities for each reduced form sign pattern with respect to each system. Accordingly, relative likelihoods for the systems can be determined based upon the Table 2 frequency distributions even if the prior probabilities of the systems and the reduced form sign pattern cannot be estimated. The generality brought forward by these results is that lower entropy systems, which provide more information about the reduced form sign patterns, will be more likely if not falsified and their acceptance will be less prone to type 2 error.

IV. Conclusion. The purpose of this paper is to demonstrate the potential of a qualitative analysis in evaluating econometric models. The use of qualitative methods can be compelling to the degree to which the theory does not usually provide a great deal of information beyond a system's inference structure and the directions of influence among its variables. Unlike the bulk of the literature on such analyses, we show that insightful, and in the case of falsification, decisive, results are always possible and do not depend upon unusually restrictive or uncommon features of the mathematical content of a model. Indeed, a qualitative analysis can enable the information content of a model to be estimated, based upon the entropy of the distribution of possible reduced form sign patterns that corresponds to the hypothesized structural sign patterns. As a model's information content is greater, i.e., as it imposes greater limits on the possible outcomes of the reduced form sign patterns, the model's falsification is more subject to type 1 error and the model's acceptance is less subject to type 2 error.

A qualitative analysis is proposed as a complement to other econometric methods that depend upon more quantitative features of the data to be analyzed. Nevertheless, in the particular case of an estimated reduced form's sign pattern falsifying the zero restrictions of an hypothesized structural form, the estimation of the structural arrays with the zero restrictions imposed will always lead to a mistake, i.e., the reduced form sign pattern is impossible for estimated structural arrays with the zero restrictions imposed. The algorithmic methods used in illustrating our points are indicative and not definitive. Significant issues with the distributional rules used for sampling the quantitative values of the entries of the structural arrays, sample size, and perhaps other features of what was done deserve considerable attention that is beyond the scope of this paper. It is hoped that such matters are addressed, since qualitative methods can significantly improve the manner in which the validity of econometric models are assessed, given the limitations of their theoretical derivation.

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Appendix: A Qualitative Tool Kit. The Monte Carlo approach we utilize in this paper was first presented in Buck and Lady (2011). We have added a number of enhancements to the method first presented as discussed below. Algorithmic approaches and the necessary conditions for conducting a qualitative analysis presented previously include: Lancaster (1966); Bassett, Maybee and Quirk (1968); Ritschard (1983); Maybee (1986); and Lady (1993). These all were designed to detect and resolve any entries in the reduced form that were signable, i.e., had the same sign, given the sign patterns of the structural arrays, but independent of the magnitudes of the nonzero entries. The algorithms were computer-based, were to one degree or another difficult to formulate based upon their textual descriptions, and (so far as we know) were not, and are not, widely utilized.

The Monte Carlo algorithm used here undertakes repeated sampling of the magnitudes of the entries of the structural arrays, consistent with the specified sign patterns. Then, $\pi = \beta^{-1}\gamma$ is computed. Given this, it is a simple procedure to count the number of times each entry of π is positive and the number of times negative (if zero the sample is discarded). This is an extremely robust method to determine if any of the entries are signable, since if so they will always turn out to have the same sign, regardless of the size of the sample. Further, these simple counts will also reveal if entries of π always have the same, or different signs, as dictated by entries of β's adjoint being signable, since if so the counts of such will be the same, independent of the size of the sample. If the sign pattern of an estimated reduced form is in-hand, it is additionally straight-forward to see if the sign patterns of any of the sampled reduced forms are the same as the estimated reduced form. The size of the arrays are not particularly limiting for this "simple search," except for the time needed to construct samples of a given size. If the sign pattern of the estimated reduced form is not found due to repeated sampling, then the counts of positive and negative entries in the sampled reduced form may reveal the reason, i.e., due to signable entries or entries required to always have the same or different signs. Typically (unfortunately), the sampled reduced forms will not provide these regularities and call for further analysis if the sign pattern of the estimated reduced form is not found, suggesting that the structural hypothesis is falsified.

Processing the algebra of the computation $\pi = \beta^{-1}\gamma$ is one of several obvious features of the analysis that calls for substantial development and algorithmic support. One immediate technique is to check to see if subgroupings of the reduced form sign pattern are resulting in the apparent falsification. This is accomplished by only checking such a subgrouping when comparing the sampled reduced forms to the estimated reduced form. When such subgroupings are identified, then the algebra of computing the reduced form can be written out with the appropriate focus to determine the problem of satisfying the systems (5). This was done in Buck and Lady (2011) and Lady and Buck (2012).

The method used here to enumerate all possible reduced form sign patterns, the associated frequency distribution in the sample taken, and the subsequent computation of the entropy of the structural hypothesis is sensitive to the size of the system. For a given system, barring zeros as we assume, there are 2^{mn} possible reduced form sign patterns. The (long) integer used in our computing platform is limited to +/- 2^{31}. Accordingly, we cannot tabulate indexed counts of the reduced form sign patterns except for $mn \leq 30$. This limitation can be mitigated by using other computing platforms or indexing schemes.

The distributions from which the quantitative values of the structural arrays are chosen can be set for each entry independent of the others, always set at a particular cardinal value as might be the case for accounting equations in the structural system, or otherwise proposed by the theory, e.g., the marginal propensity to consume is not only positive, but also might be limited to be less than one. Further, some entries can be set at different signs as well as different values. In our sampling procedures, the following notation is used for the sign of an entry in the structural arrays: (1, -1, 0) for respectively (+, -, 0), 2 for equally probable + or -, 3 for equally probable (+, -, 0), 4 for equally probable + or 0, and 5 for equally probable - or 0. When testing the zero restrictions of the structural arrays, all nonzeros are assigned a “2.” In the example using systems #1 and #2 in section III, the structural arrays specifying the “universe” of 512 possible structural sign patterns was specified as:

$$\text{sgn } \beta = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 4 & 1 & 1 & 4 \\ 1 & 4 & 1 & 1 \\ 5 & 0 & 1 & 1 \end{bmatrix} \; and \; \gamma = \begin{bmatrix} 1 & 5 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using this convention, these arrays were sampled and the corresponding reduced forms computed 300,000,000 times. The counts were tabulated for each time one of the twenty-eight reduced form sign patterns at issue was generated. The frequency distributions of these counts were then used in equation (8) for the prior probability of each of the sign patterns for π. Tables A1 and A2 below give the workup for computing the conditional probabilities for each system provided in Table 3.

Table A1: Conditional Probability for System #1

sgn π **Base 10 Index**	**P(sgn** π)	**P(Sys#1)**	**P(sgn** π|Sys #1)	**P(Sys#1/sgn** π)
8914	.00452	.00195	0.00103	.00044
8915	.00764	.00195	0.00055	.00014
9878	.00085	.00195	0.00082	.00188
9879	.00278	.00195	0.00121	.00085
9938	.00141	.00195	0.00078	.00108
9939	.01481	.00195	0.00153	.00020
21925	.01304	.00195	0.00064	.00010
26262	.00241	.00195	0.00360	.00291
26263	.00831	.00195	0.00220	.00052
38189	.00040	.00195	0.00057	.00278
38249	.00309	.00195	0.00029	.00018
39273	.00236	.00195	0.00267	.00221
40237	.00062	.00195	0.00081	.00255
40297	.00129	.00195	0.00143	.00216
41562	.00229	.00195	0.00019	.00016
41563	.00527	.00195	0.00035	.00013
41682	.00265	.00195	0.00038	.00028
41683	.01232	.00195	0.00057	.00009
42586	.00227	.00195	0.00096	.00082
42587	.00740	.00195	0.00256	.00067
42646	.00520	.00195	0.00174	.00065
42647	.01549	.00195	0.00351	.00044
42706	.00247	.00195	0.00261	.00206
42707	.02324	.00195	0.00880	.00074
43610	.00305	.00195	0.00068	.00043
43611	.00478	.00195	0.00056	.00023

54573	.00544	.00195	0.00043	.00015
56621	.00821	.00195	0.00239	.00057

Table A2: Conditional Probability for System #2

sgn π **Base 10 Index**	**P(sgn** π)	**P(Sys#2)**	**P(sgn** π|Sys #2)	**P(Sys#2/sgn** π)
8914	.00452	.00195	0.02815	.01214
8915	..00764	.00195	0.02339	.00597
9878	.00085	.00195	0.00568	.01303
9879	.00278	.00195	0.00935	.00656
9938	.00141	.00195	0.00491	.00679
9939	.01481	.00195	0.02828	.00372
21925	.01304	.00195	0.04850	.00725
26262	.00241	.00195	0.01702	.01377
26263	.00831	.00195	0.01424	.00334
38189	.00040	.00195	0.0063	.03071
38249	.00309	.00195	0.01225	.00773
39273	.00236	.00195	0.02578	.02130
40237	.00062	.00195	0.00291	.00915
40297	.00129	.00195	0.0055	.00831
41562	.00229	.00195	0.01478	.01259
41563	.00527	.00195	0.03311	.01225
41682	.00265	.00195	0.01436	.01057
41683	.01232	.00195	0.00961	.00152
42586	.00227	.00195	0.00675	.00580
42587	.00740	.00195	0.05516	.01454
42646	.00520	.00195	0.06068	.02276
42647	.01549	.00195	0.03283	.00413
42706	.00247	.00195	0.01609	.01270
42707	.02324	.00195	00.02511	.00211
43610	.00305	.00195	0.02359	.01508
43611	.00478	.00195	0.02556	.01043
54573	.00544	.00195	0.01887	.00676
56621	.00821	.00195	0.01847	.00439