Nonaggregable evolutionary dynamics under payoff heterogeneity

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Department of Economics
DETU Working Paper 17-02
October 2017

1301 Cecil B. Moore Avenue, Philadelphia, PA 19122

http://www.cla.temple.edu/economics/faculty/detu-working-paper-series/

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October 8, 2017

Abstract

We consider general evolutionary dynamics under idiosyncratic but persistent payoff heterogeneity and study the dynamic relation between the strategy composition over different payoff types and the aggregate strategy distribution of the entire population. It is rigorously proven that continuity of the switching rate function or the type distribution guarantees the existence of a unique trajectory. In major evolutionary dynamics except the standard best response dynamic, an agent's switching rate from the current action to a new action increases with the payoff gain from this switch. This payoff sensitivity makes a heterogeneous dynamic nonaggregable: the transition of the aggregate strategy generically depends not only on the current aggregate strategy but also on the current strategy composition. However, if we look at the strategy composition, stationarity of equilibrium in general and stability in potential games hold under any admissible dynamics. In particular, local stability of each individual equilibrium composition under an admissible dynamic is equivalent to that of the corresponding aggregate equilibrium in the aggregate dynamic induced from the standard best response dynamic, though the basin of attraction may differ over different dynamics.

Keywords: evolutionary dynamics, payoff heterogeneity, aggregation, continuous space, potential games

JEL classification: C73, C62, C61.

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1 Introduction

In a population game, a population of (infinitely or finitely many) agents plays a game; each agent takes the role of a ”player” in a normal form of the game. While it is commonly assumed that the agents in the population are homogeneous,[1] there are a few studies that bring payoff heterogeneity into the game and discuss the relation between aggregation and dynamics: Blonski (1999) and Ely and Sandholm (2005). But, these studies rely on the *aggregability* of the dynamic—the change in the aggregate strategy distribution is wholly determined from the current state of the aggregate distribution alone, independent of the underlying correlation between strategy choices and payoff types. Such aggregability may be assumed as in Blonski (1999) or may be derived from some specific form of the agents' strategy revision processes as in Ely and Sandholm (2005). In this paper, we consider a general class of evolutionary dynamics in the heterogeneous setting without requiring aggregability. The ultimate goal of this paper is to provide a foundation for nonaggregable dynamics that would better capture the dynamic relations between heterogeneous microscopic behavior and the macroscopic aggregate state beyond the representative agent approach.

Aggregation in evolutionary dynamics. Generally, in an evolutionary dynamic, an agent occasionally switches its action. The switching decision of an agent is supposed to follow a *switching rate function* (revision protocol), which determines the switching rate from one action to another based on the payoff vector and (possibly, but not necessarily) the action distribution over the observed population of other agents. The population dynamic is obtained by summing the switching processes of individual agents.

In a heterogeneous setting, there are two layers to describe the distribution of actions over the heterogeneous population. *Strategy composition* is the joint distribution of actions and types, while the *aggregate strategy* is its marginal distribution of actions, i.e., the distribution of actions collected over all the agents regardless of their types. We consider aggregate games with payoff heterogeneity: the payoff of an action for each agent not only changes with aggregate strategy but also differs depending on the type of agent. If the choice of a new action is solely based on the payoffs and not on the other agents' action distribution as in the best response dynamic and payoff comparison dynamics, an individual agent's switching rate is completely determined in an aggregate game only from the aggregate strategy and the agent's own type without identifying the strategy composition.

However, different types of agents may have different switching rates. The population dynamic is driven by switching agents. Thus, to pin down the population dynamic, we need to identify which agents are more likely to switch actions, namely, which agents have greater switch-

[1] Of course, it is very common in evolutionary game theory to have multiple populations, each of which represents a different player in the normal form and consists of homogeneous agents. Our theorems on the stationarity and stability of equilibrium composition can be seen as an extension of the stationarity and stability of a Nash equilibrium in finitely many populations to (potentially) continuously many populations; however, our extension comes straight from those properties in a single population setting. After all, our motivation and the central issue in this paper is the relationship between *aggregation* and dynamics.

ing rates. The transition of the aggregate strategy is more dictated by the switches of those agents. For example, suppose that the population is divided into two groups of agents with equal masses; switching rates for agents of one group are much greater than those for the other group, which are relatively close to zero. Then, the transition of the aggregate strategy is determined mostly from the transition of the latter group's action distribution, as illustrated in Figure 1. When the switching rate depends on the payoff gain from the switch as in most evolutionary dynamics, this identification problem reduces to the identification of which agents face greater payoff gains from switches. Under payoff heterogeneity, the payoff gains for agents vary not only with their current actions but also with their types. Therefore, the transition of the aggregate strategy generally depends not only on the distribution of current actions alone, i.e., the current aggregate strategy, but also on the joint distribution of types and actions, i.e., the strategy composition.

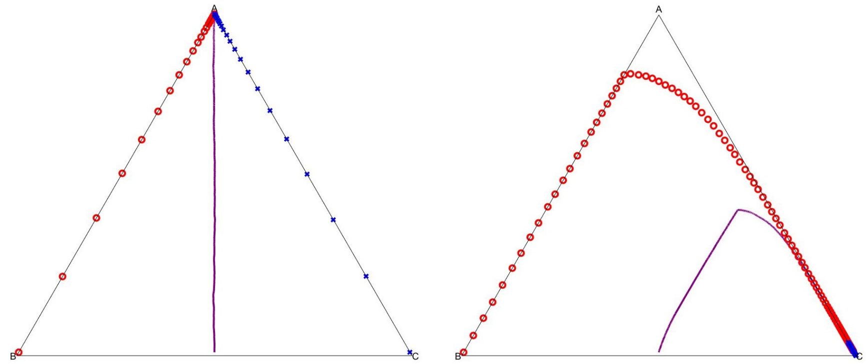
Aggregable dynamics. Ely and Sandholm (2005) consider the standard best response dynamic (BRD) in a population of heterogeneous agents. In their *heterogeneous standard BRD*,[2] every switching agent chooses the best response action that currently yields the greatest payoff for the agent, and the switching rate to the best response action is constant and common for all agents. Agents may currently take different actions and face different amounts of payoff gains from switches; every agent will switch to the best response action for its own type at the constant and common rate. Thus, the transition of the aggregate strategy is determined solely by filling the gap between the distribution of current actions and that of the best-response actions over all agents in the society; the gap diminishes at a constant rate, regardless of the difference in payoff gains among these agents. Thus, the heterogeneous standard BRD is *aggregable*, in the sense that the transition of the aggregate strategy is completely determined by the current state of the aggregate strategy alone. Correlation between types and actions does not matter for the transition of the aggregate strategy *at all*.

Aggregability eases the analysis significantly, as it reduces the dimension of the dynamic: while the strategy composition $\mathbf{X}$ is a joint distribution over the product space $\mathcal{A} \times \Theta$ of actions and payoff types, the aggregate strategy $\bar{\mathbf{x}}$ is just a distribution over action set $\mathcal{A}$. But this also suggests that we cannot employ such an aggregable dynamic, to discuss for example how the correlation of choices and incentives at the microscopic agent level dynamically affects the macroscopic aggregate state beyond a representative agent approach. Besides, it is natural to expect that a huge disbalance in the incentives to switch actions among different types would cause volatility of the social state. For the study of these issues, nonaggregable dynamics would be a more appropriate choice as agents' switching behavior is more consistent with payoff heterogeneity in such dynamics. However, no preceding research studies the relationship between aggregation and evolutionary dynamics beyond the aggregable (i.e., representative agent) framework.

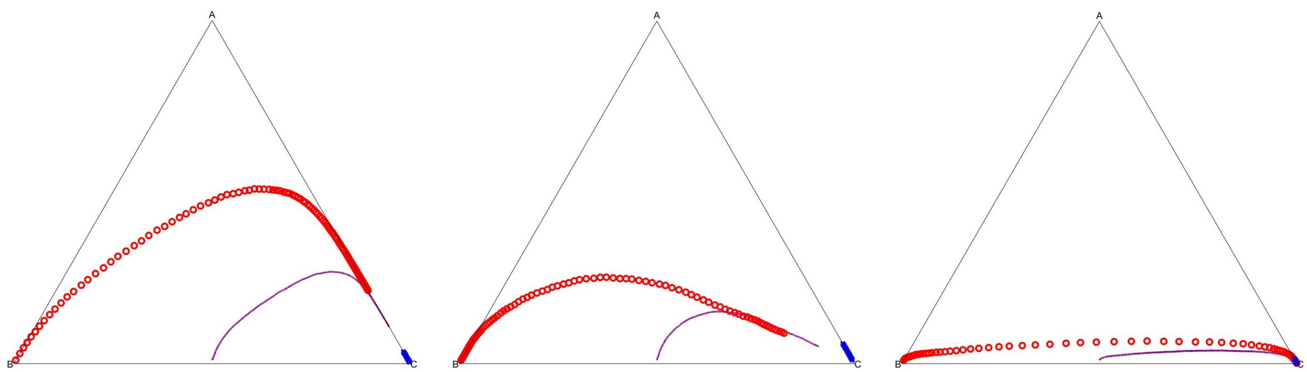
[2] In their paper, the heterogeneous standard BRD is called Bayesian BRD, and the homogenized smooth BRD as the aggregate dynamic of the heterogeneous standard BRD is called the aggregate BRD.

	A	B	C
A	$1+\theta$	$1+\theta$	$1+\theta$
B	1	2	0
C	1	0	2

(a) Payoff table for type θ



(b) Standard BRD (c) Tempered BRD



(d) Smith (e) BNN (f) Replicator

A solid purple line indicates the trajectory of aggregate strategy $\bar{\mathbf{x}}$, and sequences of red circles and blue crosses show the trajectories of Bayesian strategies $\mathbf{x}(\theta^H)$ and $\mathbf{x}(\theta^L)$. These trajectories are drawn from agent-based simulations of discrete-time dynamics with 10000 agents of each type for 5000 periods; an agent receives a revision opportunity with probability 0.005 in each period. In each dynamic except the standard BRD, the switching rate (conditional on the receipt of a revision opportunity) is basically set to $1/2$ of the payoff gain from the revision. (Note that the maximal payoff difference is $2-0$ and thus the conditional switching rate is at most 1.)

Figure 1: Dynamics of aggregate strategy in a symmetric 3-action coordination game. The population is divided to equal masses of two types, $\theta^H = 0.4$ and $\theta^L = 0.1$. The initial composition is set to $\mathbf{x}_0(\theta^H) = (\varepsilon, 1-\varepsilon, 0)$ (near $\mathbf{e}^B = (0,1,0)$, the left bottom corner of a Kolm triangle) and $\mathbf{x}_0(\theta^L) = (\varepsilon, 0, 1-\varepsilon)$ (near $\mathbf{e}^C = (0,0,1)$, the right bottom corner), with $\varepsilon = 0.01$. In most dynamics except the standard BRD, the aggregate dynamic is more driven by the dynamic of a Bayesian strategy of type θ^H than that of type θ^L. Type θ^H indeed has a greater payoff gain from revisions, especially around initial periods, and thus has a greater switching rate except in the standard BRD, in which the switching rate is constant. The asymmetry of trajectories (curved toward $\mathbf{e}^C$) suggests the dependency of these dynamics on the initial strategy composition. See the Supplementary Note for a more detailed analysis of this example and the trajectories when the initial composition is reversed between type θ^H and type θ^L.

Generic nonaggregability of evolutionary dynamics. Many other dynamics such as pairwise comparison dynamics, tempered BRDs, excess payoff dynamics and imitative dynamics have the switching rate dependent on the amount of payoff gain from the switch. This creates biases in the action distribution of *switching* agents compared to that of all the other agents; the transition of the aggregate strategy is more influenced by those who have greater payoff gains. So it matters which agents face greater payoff gains; the transition of the aggregate strategy cannot be predicted without identifying the underlying strategy composition.

In Theorem 2, we rigorously prove that, if different types of agents have different switching rates, the transition of the aggregate strategy varies with the current strategy composition, not only with the current aggregate strategy: that is, the dynamic is not aggregable. The nonaggregability condition in this theorem holds for all the above mentioned dynamics. Furthermore, in Theorem 3, we consider a binary coordination game and find that, even if an aggregate equilibrium is stable under the standard BRD and thus the corresponding equilibrium composition is stable under any admissible dynamic, the aggregate strategy may escape from this aggregate equilibrium under a nonaggregable admissible dynamic. This happens when the underlying initial strategy composition largely differs from the equilibrium composition that aggregates to the stable equilibrium and the variation in the switching rate is sufficiently large. These non-aggregability theorems suggest that an aggregable dynamic may fail to predict not only short-run transitions but also long-run outcomes, unless agents' switching rates are invariant to the degree of payoffs.

Stationarity and stability of equilibrium compositions. In contrast to these negative results on aggregability, we verify that the stationarity and stability of Nash equilibria extend to those of equilibrium *compositions* in the heterogeneous setting. These results have been established in the homogeneous setting for quite a wide class of evolutionary dynamics: see Sandholm (2010). For this, the least demanding properties are assumed based on the consistency between payoffs and switches in evolutionary dynamics. First, no agents should switch actions if and only if they are choosing the best response actions to their current payoff vector (best response stationarity, Definition 3). Secondly, the net change in the mass of each action's players should always be positively correlated with the relative payoff of the action (the positive correlation, Definition 6). We call a dynamic that satisfies these two properties an *admissible* dynamic. We extend equilibrium stationarity in general and stability in potential games under admissible dynamics to the heterogeneous setting (Theorems 4 and 5).

We obtain these positive results by directly analyzing the dynamic of strategy composition over payoff types. The dynamic of the strategy composition over types is defined as a differential equation over the space of probability measures on the product space of actions and types. We rigorously define it from the individual agents' revision processes and verify the existence of a solution path under mild continuity conditions in Theorem 1. For this rigorous formulation of the dynamic over a (possibly) continuous-*type* space in our model, we borrow the formulation and techniques from the recent literature on continuous-*strategy* evolutionary dynamics, especially Oechssler and Riedel (2001, 2002) and Cheung (2014).[3] Note that Milgrom and Weber (1985) consider strategy composition in a general incomplete information game as a joint probability measure over types and actions, which they call *distributional strategy*, and verify fundamental properties of equilibrium distributional strategy such as existence and purification. The measure-

[3]To name a few more, see also Hofbauer, Oechssler, and Riedel (2009), Friedman and Ostrov (2013), Lahkar and Seymour (2013) and Lahkar and Riedel (2015).

theoretic formulation of heterogeneous dynamics in this paper provides a rigorous foundation for the evolutionary dynamics of distributional strategy.

Specifically for potential games, we prove global asymptotic stability of the set of equilibrium compositions in Theorem 5. We further find in Corollary 2 that the equivalence between a local maximum of the potential game and a locally stable equilibrium holds for any admissible heterogeneous dynamic, which includes the standard BRD. The aggregate dynamic —the *homogenized smooth BRD*—is constructed by having a transitory payoff perturbation instead of a persisitent payoff perturbation. Each agent's idiosyncratic payoff can change over time and follows a common i.i.d. process, which makes all agents homogeneous. To obtain the aggregate dynamic of a heterogeneous standard BRD, the transitory idosyncratic payoff is supposed to follow the same probability distribution as that of permanent idiosyncratic payoffs in the heterogeneous setting. The aggregability theorem in Ely and Sandholm (2005) states that the aggregate strategy under the heterogeneous standard BRD completely follows the homogenized smooth BRD. Combined with their aggrebability result on the standard BRD, our local stability theorem suggests that the local stability of an equilibrium under any heterogeneous dynamic can be tested just by examining the local stability of the corresponding aggregate equilibrium under the homogenized smooth BRD: despite generic nonaggregability, the local stability of each equilibrium does not change by nonaggregability when we consider a potential game in admissible dynamics.

Yet, the homogenized smooth BRD may fail to predict the transition and long-run outcome from a *specific* initial state, even in a potential game. The fallacy lies in the gap between the topologies of strategy composition and aggregate strategy. Local stability under a non-aggregable dynamic may require the initial *composition* to be close to the equilibrium composition, while local stability under a homogenized smooth BRD requires only the initial *aggregate* strategy to be close to the aggregate equilibrium. We clarify this gap by assuming additive separability of payoff heterogeneity and then comparing a potential function for admissible dynamics (the *heterogeneous* potential function) with a Lyapunov function for the homogenized smooth BRD (the *homogenized* potential function). Under additive separability, the payoff of each action is decomposed to the common payoff function and the idiosyncratic payoff constant: the former depends on the current aggregate strategy but is common to all agents, and the latter depends only on the type of agent but does not change with the aggregate strategy or strategy composition. To make a potential game, the common payoff function is supposed to have a potential function (the *original* potential function). The heterogeneous potential function is defined on the space of strategy composition by adding a negative entropy term that accounts for the sortednesss of the current strategy composition to the original potential function, while the homogenized potential function is defined on the space of aggregate strategy by adding the expected idiosyncratic payoffs of homogenized agents to the original potential. We find that the latter serves as an upper bound of the former and they coincide if and only if the strategy composition is at an equilibrium; the heterogeneous potential must increase over time under admissible dynamics and the homogenized potential must increase under the homogenized smooth BRD. So one may expect them to behave similarly under a dynamic. However, the increase in the heterogeneous potential may owe much to the increase in the negative entropy term (the decrease in entropy) due to sorting pressure. In this case, the original potential may not increase; then, neither does the homogenized potential. Thus, the aggregate strategy may move away from the closest local maximum of the homogenized potential function. Therefore, the transition of the aggregate strategy under a nonaggregable dynamic may completely differ from that of the homogenized smooth BRD.

Implications on empirical and applied study. In an empirical work on discrete choice, an economist may not have access to micro data and thus may need to use coarse aggregate data. Our negative results on generic nonaggregability suggests that, even if the economist could precisely identify the underlying payoff structure, the sensible difference between the transitory payoff perturbation and the persistent payoff heterogeneity results in qualitatively different predictions of the aggregate dynamic. This raises concerns about empirical studies with aggregate data and puts a limitation on the interpretation of heterogeneity accounted for by such empirical studies. As we argue in Section 4, our concern accords with the econometric concern on sample selection raised by Artuç, Lederman, and Porto (2015); we will see that the problem caused by a wrong specification of heterogeneity cannot be resolved even if there were no bias in the estimation.

On the other hand, our positive results on asymptotic equivalence of local stability can relieve the concerns of applied economists, as long as their main focuses are on long-run outcomes in potential games. Suppose that their models have only one stable aggregate equilibrium when they test the dynamic by assuming only transient payoff heterogeneity, i.e., by employing the homogenized smooth BRD. Corollary 2 tells us that the local stability of the aggregate equilibrium is retained over all admissible dynamics even if the payoff heterogeneity is persistent.

For a practical application, consider the dynamic implementation of a social optimum in a congestion game. In the homogeneous setting, a desirable aggregate state could be achieved by a very simple bang-bang control that gives a subsidy for the actions that need more players and imposes a tax on the actions that need less. By keeping the taxes and subsidies at extreme levels in their feasible ranges, convergence can be achieved in a finite time; and it is the fastest among all the tax/subsidy schemes. But, in the heterogeneous setting, such extreme pricing may result in excessive distortion of the underlying composition and practically unacceptable instability. To avoid this, pricing should be less extreme and adjusted continuously over time. Actually, our stability theorem implies that the dynamic Pigouvian pricing, proposed by Sandholm (2002, 2005), assures convergence to the social optimum, as long as agents' switching behavior is consistent with the current payoffs as generally assumed for the admissibility of evolutionary dynamics. It does not require the social planner to know the underlying strategy composition or to identify the underlying dynamic, but works perfectly.

Outline of the paper. The paper proceeds as follows. In the next section, we define the game under payoff heterogeneity, paying attentions to the distinction between the *aggregate strategy* and the *strategy composition*; we build a heterogeneous evolutionary dynamic from an individual agent's switching rate function. In Section 3, we verify the Lipschitz continuity of this dynamic in order to well define the dynamic and guarantee the existence of a unique solution path. In Section 4, we formally define the aggregability of heterogeneous evolutionary dynamics and argue the generic nonaggregability of heterogeneous dynamics. Section 5 is devoted to presenting the positive results on equilibrium composition: we extend the stationarity of a Nash equilibrium and its stability in potential games to that of equilibrium compositions in heterogeneous dynamics; further, we study the local stability of each equilibrium composition in potential games and discuss its implications and applications. Until this section, we focus on non-observational evolutionary dynamics in which an agent's switching rate depends only on the payoff vector for the agent but not on the other agents' actions. In Section 6, we consider observational dynamics such as imitative dynamics and excess payoff comparison dynamics and argue that the theorems in this paper are readily applied to them as long as each agent observes only agents of the same type; we discuss the case where an agent observes the aggregate strategy of the entire society instead. Section 7 wraps up the paper. Appendices provide the proofs and a few technical details on the measure-theoretic construction of heterogeneous dynamics. Parts of proofs that essentially only involve heavy calculation are found in the Supplementary Note.

2 Model

2.1 Aggregate games with payoff heterogeneity

Consider a large population of agents $\Omega := [0,1] \subset \mathbb{R}$ who share the same action set $\mathcal{A} = \{1, \cdots, A\}$. We define probability measure $\mathbb{P}_\Omega : \mathcal{B}_\Omega \to [0,1]$ as Lebesgue measure so $\mathbb{P}_\Omega(\Omega) = 1$. Denote by $\mathcal{B}_\Omega$ the Lebesgue σ-field over Ω. Denote by $\mathfrak{a}(\omega)$ the action taken by agent $\omega \in \Omega$. We restrict action profile $\mathfrak{a} : \Omega \to \mathcal{A}$ to a $\mathcal{B}_\Omega$-measurable function. Then, $\bar{x}_a := \mathbb{P}_\Omega(\{\omega \in \Omega : \mathfrak{a}(\omega) = a\}) \in [0,1]$ is the mass of agents who take action $a \in \mathcal{A}$. We call $\bar{\mathbf{x}} := (\bar{x}_a)_{a \in \mathcal{A}} \in \Delta^A$ the **aggregate strategy**, where $\Delta^A := \{\mathbf{z} \in \mathbb{R}^A_+ : \sum_{a \in \mathcal{A}} z_a = 1\}$ is the set of A-dimensional probability vectors.

We focus on **aggregate games** with payoff heterogeneity, as follows. Each agent $\omega \in \Omega$ is assigned to type $\boldsymbol{\theta}(\omega) \in \mathbb{R}^T$. Then, the agent's payoff from action a is $F_a[\bar{\mathbf{x}}](\boldsymbol{\theta}(\omega))$ when the aggregate state is $\bar{\mathbf{x}}$. Thus, $\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta}) \in \mathbb{R}^A$ is the payoff vector at aggregate state $\bar{\mathbf{x}}$ for type $\boldsymbol{\theta}$. Let $|\cdot|^\infty_T$ be the sup norm on $\mathbb{R}^T$ and $\mathcal{B}_\Theta$ be the Borel σ-field on this metric space $\mathbb{R}^T$.[4] Agents' type profile $\boldsymbol{\theta} : \Omega \to \mathbb{R}^T$ is assumed to be measurable with respect to $\mathcal{B}_\Omega$. Then, it induces probability measure $\mathbb{P}_\Theta : \mathcal{B}_\Theta \to [0,1]$ by $\mathbb{P}_\Theta(B_\Theta) := \mathbb{P}_\Omega(\boldsymbol{\theta}^{-1}(B_\Theta))$ for each $B_\Theta \in \mathcal{B}_\Theta$. Denote by $\Theta \subset \mathbb{R}^T$ the support of $\mathbb{P}_\Theta$; we call it the type space. Given $\bar{\mathbf{x}}$, $\mathbf{F}[\bar{\mathbf{x}}] : \Theta \to \mathbb{R}^T$ is assumed to belong to $\mathcal{C}_\Theta$, the set of $\mathcal{B}_\Theta$-measurable continuous functions from Θ to $\mathbb{R}^A$.

In contrast to the aggregate strategy, we define **Bayesian strategy** $\mathbf{x} = (x_a)_{a \in \mathcal{A}} : \Theta \to \Delta^A_+$ to represent the strategy composition of actions and types, following terminology of Ely and Sand-

[4] For a vector $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_T) \in \mathbb{R}^T$, the sup norm of the vector is $|\boldsymbol{\theta}|^\infty_T = \max\{|\theta_1|, \ldots, |\theta_T|\}$. (We omit the super/subscripts when it is obvious.)

holm (2005).[5] More specifically, $x_a(\boldsymbol{\theta})$ is the *proportion* of action-a players among subpopulation of type-$\boldsymbol{\theta}$ agents. For example, our formulation allows $\mathbb{P}_\Theta$ to have a finite support; then, the Bayesian strategy is obtained as

$$x_a(\boldsymbol{\theta}) = \mathbb{P}_\Omega\left(\{\omega \in \Omega : \mathfrak{a}(\omega) = a \text{ and } \boldsymbol{\theta}(\omega) = \boldsymbol{\theta}\}\right) / \mathbb{P}_\Theta(\boldsymbol{\theta})$$

for each type in the support of $\mathbb{P}_\Theta$. The aggregate strategy is expressed in terms of the Bayesian strategy as[6]

$$\bar{x}_a = \mathbb{E}_\Theta x_a, \qquad \text{i.e., } \bar{\mathbf{x}} = \mathbb{E}_\Theta \mathbf{x}.$$

In general, we first define **strategy composition** $\mathbf{X} : \mathcal{B}_\Theta \to \Delta^A$ as the joint distribution of actions and types such that the marginal distribution of types coincides with $\mathbb{P}_\Theta$. Then, Bayesian strategy $\mathbf{x} : \Theta \to \Delta^A$ is defined as its Radon-Nikodym density such as $X_a(B_\Theta) = \int_{B_\Theta} x_a d\mathbb{P}_\Theta$ for each $a \in \mathcal{A}, B_\Theta \in \mathcal{B}_\Theta$; we abbreviate this relationship as $\mathbf{X} = \int \mathbf{x} d\mathbb{P}_\Theta$. Denote by $\mathcal{F}_\mathcal{X}$ the set of Bayesian strategies.[7] Similarly, let $\mathcal{X}$ be the set of strategy compositions. Due to one-to-one correspondence between $\mathbf{X}$ and $\mathbf{x}$, they can be seen as equivalent. We consider Bayesian strategies in the main body of this paper for simplicity in expositions and appealing to intuition, though we prove most theorems by arguing strategy composition for technical reasons.[8] See Appendix A.1 for the measure-theoretic formulation of the model and dynamic.

Example 1 (Additively Separable Aggregate game (ASAG)). In the context of discrete choice models such as in Anderson, De Palma, and Thisse (1992), it is common to introduce payoff heterogeneity in an additively separable manner. That is, the payoff function is additively separated to the common part and the idiosyncratic part: with $T = A$, type $\boldsymbol{\theta} = (\theta_a)_{a\in\mathcal{A}} \in \mathbb{R}^A$ is defined as a vector of the idiosyncratic payoff of each action for this type, which varies among agents but does not change over time regardless of the state of the society. Then, an agent of this type receives payoff $F_a[\bar{\mathbf{x}}](\boldsymbol{\theta}) = F_a^0(\bar{\mathbf{x}}) + \theta_a$ by taking action a. Given the aggregate state $\bar{\mathbf{x}}$, $\mathbf{F}^0(\bar{\mathbf{x}}) = (F_a^0(\bar{\mathbf{x}}))_{a\in\mathcal{A}} \in \mathbb{R}^A$ is the common payoff vector, shared by all the agents in the society Ω. Thus, for each $\bar{\mathbf{x}} \in \Delta^A$ and

[5] When calling $\mathbf{x} : \Theta \to \Delta_+^A$ a Bayesian strategy, we would imagine a Bayesian game where a player chooses a strategy (a contingent action plan) before he knows his own type. In a Bayesian game, we distinguish a 'player' and an 'agent.' A player comes to the game before knowing its type, and decides on a plan of the action contingent on the realized type: a Bayesian strategy is this contingent plan of one player. In a Bayesian *population* game, an agent comes to the game after knowing his type and decides on an action; the Bayesian strategy is essentially equivalent to an empirical joint distribution of type and actions.

[6] Here $\mathbb{E}_\Theta$ is the expectation operator on the probability space $(\Theta, \mathcal{B}_\Theta, \mathbb{P}_\Theta)$, while $\mathbb{E}_\Omega$ is that on $(\Omega, \mathcal{B}_\Omega, \mathbb{P}_\Omega)$: i.e., $\mathbb{E}_\Omega f := \int_\Omega f(\omega)\mathbb{P}_\Omega(d\omega)$ for a $\mathcal{B}_\Omega$-measurable function $f : \Omega \to \mathbb{R}$ and $\mathbb{E}_\Theta \tilde{f} := \int_\Theta \tilde{f}(\boldsymbol{\theta})\mathbb{P}_\Theta(d\boldsymbol{\theta})$ for a $\mathcal{B}_\Theta$-measurable function $\tilde{f} : \Theta \to \mathbb{R}$. If $f = \tilde{f} \circ \boldsymbol{\theta}$, then we have $\mathbb{E}_\Theta \tilde{f} = \mathbb{E}_\Omega f$.

[7] Two Bayesian strategies $\mathbf{x}, \mathbf{x}' \in \mathcal{F}_\mathcal{X}$ are considered as identical, i.e., $\mathbf{x} = \mathbf{x}'$ if $\mathbf{x}(\boldsymbol{\theta}) = \mathbf{x}'(\boldsymbol{\theta})$ for $\mathbb{P}_\Theta$-almost all $\boldsymbol{\theta} \in \Theta$. They indeed yield the same strategy composition.

[8] For mathematical construction of the dynamics and analysis of stability, we basically follow the literature on evolutionary dynamics on continuous strategy space, such as Oechssler and Riedel (2001) and Cheung (2014). When proving the existence of a unique solution path, the state space of the dynamic needs to be a Banach space. For this, they first define the dynamic as a dynamic of probability measure (over the continuous strategy space in theirs and over the type-action space in ours) and then extend the state space of the dynamic from the probability measure space to the space of finite signed measures; we take this approach in Appendix A.3. For stability analysis, we use the Lyapunov stability theorem as in Theorem 10 in Appendix C, in which the stability concept is defined for the weak topology on $\mathcal{X}$.

$\boldsymbol{\theta} \in \mathbb{R}^A$,

$$\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta}) := \mathbf{F}^0(\bar{\mathbf{x}}) + \boldsymbol{\theta} \in \mathbb{R}^A \tag{1}$$

is the payoff vector for type $\boldsymbol{\theta}$ when the aggregate strategy is $\bar{\mathbf{x}}$. We call an aggregate game with such additively separable idiosyncratic payoffs an **additively separable aggregate game (ASAG)**. ▮

Example 2 (Binary aggregate game). When arguing an aggregate game with two actions, we commonly denote the action set as $\mathcal{A} = \{I, O\}$. We can imagine an entry game in which I means participation (IN) to a certain platform and O means nonparticipation (OUT). We call such a game a **binary aggregate game**.[9] If we further assume additive separability of payoff heterogeneity, it reduces without loss of generality to the payoff function defined by $F^0_O(\bar{\mathbf{x}}) \equiv 0$ for each $\bar{\mathbf{x}} \in \Delta^2$ and $\theta_I(\omega) \equiv 0$ for each $\omega \in \Omega$. Now an agent's type $\boldsymbol{\theta}$ is identified by $\theta_O \in \mathbb{R}$ alone. We can interpret θ_O as the agent's valuation of an outside option. Denote by $P_\Theta : \mathbb{R} \to [0,1]$ be the cumulative distribution function of θ_O and $\Theta_O \subset \mathbb{R}$ be the support of θ_O. Then, $\boldsymbol{\theta} \in \Theta = \{0\} \times \Theta_0$, $\bar{\mathbf{x}} \in \Delta^2$ and $\mathbf{x} : \Theta \to \Delta^2$ are identified from $\theta_0 \in \Theta_0$, $\bar{x}_I \in [0,1]$ and $x_I : \Theta_O \to [0,1]$, respectively. We call this type of a binary aggregate game a **binary ASAG**. ▮

2.2 Bayesian equilibrium and aggregate equilibrium

Each agent's best response is determined from the aggregate strategy. Let $b[\bar{\mathbf{x}}](\boldsymbol{\theta}) \subset \mathcal{A}$ be the set of type-$\boldsymbol{\theta}$'s best response actions to aggregate strategy $\bar{\mathbf{x}}$ and $F_*[\bar{\mathbf{x}}](\boldsymbol{\theta}) \in \mathbb{R}$ be the payoff from the best response action.

$$b[\bar{\mathbf{x}}](\boldsymbol{\theta}) := \operatorname*{argmax}_{a \in \mathcal{A}} F^0_a[\bar{\mathbf{x}}](\boldsymbol{\theta}), \qquad F_*[\bar{\mathbf{x}}](\boldsymbol{\theta}) := \max_{a \in \mathcal{A}} F^0_a[\bar{\mathbf{x}}](\boldsymbol{\theta}).$$

Given aggregate strategy $\bar{\mathbf{x}}$, $b_a^{-1}[\bar{\mathbf{x}}] := \{\boldsymbol{\theta} \in \Theta : a \in \beta[\bar{\mathbf{x}}](\boldsymbol{\theta})\}$ is the set of types for which action a is a best response. Denote by $\beta_a^{-1}[\bar{\mathbf{x}}] := \{\boldsymbol{\theta} \in \Theta \ : \ b[\bar{\mathbf{x}}](\boldsymbol{\theta}) = \{a\}\} \subset \bar{\beta}_a^{-1}[\bar{\mathbf{x}}]$ the set of types for which action a is the only best response to $\bar{\mathbf{x}}$. Let $B[\bar{\mathbf{x}}](\boldsymbol{\theta})$ be the set of action distributions that assign positive probabilities only to the best response actions for type $\boldsymbol{\theta}$ given aggregate strategy $\bar{\mathbf{x}}$, i.e., the set of type-$\boldsymbol{\theta}$ agents' best response mixed strategies to $\bar{\mathbf{x}}$:

$$B[\bar{\mathbf{x}}](\boldsymbol{\theta}) := \{\mathbf{y} \in \Delta^A : y_a > 0 \ \Rightarrow \ a \in b[\bar{\mathbf{x}}](\boldsymbol{\theta})\} \qquad \text{for each } \bar{\mathbf{x}} \in \Delta^A, \boldsymbol{\theta} \in \Theta.$$

In a Nash equilibrium, (almost) every agent correctly predicts the strategy composition and takes the best response to it. Correspondingly, Bayesian strategy $\mathbf{x} \in \mathcal{F}_\mathcal{X}$ is in **Bayesian equilibrium**, if

$$\mathbf{x}(\boldsymbol{\theta}) \in B[\bar{\mathbf{x}}](\boldsymbol{\theta}) \quad \text{with } \bar{\mathbf{x}} = \mathbb{E}_\Theta \mathbf{x} \qquad \text{for } \mathbb{P}_\Theta\text{-almost all } \boldsymbol{\theta} \in \Theta, \tag{2}$$

[9] Blonski (1999) studies *aggregable* dynamics in a binary aggregate game in which $\dot{\bar{\mathbf{x}}}$ is assumed to be determined by the difference between the aggregate mass of agents for whom I is the best response to the current aggregate strategy and that of agents for whom O is the best response.

or equivalently,

$$x_a(\boldsymbol{\theta}) = \begin{cases} 1 & \text{if } \boldsymbol{\theta} \in \beta_a^{-1}[\bar{\mathbf{x}}], \\ 0 & \text{if } \boldsymbol{\theta} \notin b_a^{-1}[\bar{\mathbf{x}}] \end{cases} \quad \text{with } \bar{\mathbf{x}} = \mathbb{E}_\Theta \mathbf{x} \qquad \text{for all } a \in \mathcal{A} \text{ and } \mathbb{P}_\Theta\text{-almost all } \boldsymbol{\theta} \in \Theta.$$

That is, if a is the unique best response for type $\boldsymbol{\theta}$, (almost) all the agents of this type should take it; if a is not a best response, (almost) none of these agents should take it. We leave indeterminacy of $x_a(\boldsymbol{\theta})$ if there are multiple best response actions for $\boldsymbol{\theta}$ and a is just one of them.

Aggregation of Bayesian equilibrium over types yields

$$\mathbb{P}_\Theta(\beta_a^{-1}[\bar{\mathbf{x}}]) \leq \bar{x}_a \leq \mathbb{P}_\Theta(b_a^{-1}(\bar{\mathbf{x}})) \qquad \text{for all } a \in \mathcal{A}. \tag{3}$$

If aggregate strategy $\bar{\mathbf{x}}$ satisfies condition (3), it is called an **aggregate equilibrium**. Notice that aggregate equilibrium does *not* imply that the underlying Bayesian strategy is a Bayesian equilibrium. Bayesian equilibrium needs complete sorting of agents by types, while only the total mass of each action's players matters to aggregate equilibrium.

2.3 Construction of heterogeneous dynamics

In an evolutionary dynamic, an agent occasionally revises the action, following a Poisson process. The timing of switch and the choice of which action to switch to are determined by **switching rate function** $\mathbf{R} = (R_{ij})_{i,j \in \mathcal{A}} : \mathbb{R}^A \to \mathbb{R}_+^{A \times A}$. An economic agent should base the switching decision on the payoff vector that the agent is facing. Let $\boldsymbol{\pi}(\boldsymbol{\theta}) \in \mathbb{R}^A$ be the payoff vector for type $\boldsymbol{\theta}$. The switching rate $R_{ij}(\boldsymbol{\pi}(\boldsymbol{\theta})) \in \mathbb{R}_+$ is a Poisson arrival rate at which a type-$\boldsymbol{\theta}$ agent switches to action $j \in \mathcal{A}$ conditional on that the agent has been taking action i so far, given payoff vector $\boldsymbol{\pi}(\boldsymbol{\theta})$. The analysis in this paper is applicable to *observational dynamics*, in which switching rates also depend on the action distribution of others, but we postpone it to Section 6 for clearness of exposition.

Under switching rate function $\mathbf{R} : \mathbb{R}^A \to \mathbb{R}_+^{A \times A}$, we construct the mean dynamic of Bayesian strategy over $\mathcal{F}_\mathcal{X}$ with function $\mathbf{v} = (v_i)_{i \in \mathcal{A}} : \mathbb{R}^A \times \Delta^A \to \mathbb{R}^A$ by

$$\begin{aligned} \dot{x}_i(\boldsymbol{\theta}) = v_i(\boldsymbol{\pi}(\boldsymbol{\theta}), \mathbf{x}(\boldsymbol{\theta})) &:= \sum_{j \in \mathcal{A}} x_j(\boldsymbol{\theta}) R_{ji}(\boldsymbol{\pi}(\boldsymbol{\theta})) - x_i(\boldsymbol{\theta}) \sum_{j \in \mathcal{A}} R_{ij}(\boldsymbol{\pi}(\boldsymbol{\theta})), \\ \text{i.e., } \dot{\mathbf{x}}(\boldsymbol{\theta}) &= \mathbf{v}(\boldsymbol{\pi}(\boldsymbol{\theta}), \mathbf{x}(\boldsymbol{\theta})) \end{aligned} \tag{4}$$

for each type $\boldsymbol{\theta} \in \Theta$ and each action $i \in \mathcal{A}$. In an infinitesimal length of time $dt \in \mathbb{R}$, $\sum_{j \in \mathcal{A}} x_j(\boldsymbol{\theta}) R_{ji}(\boldsymbol{\pi}(\boldsymbol{\theta}))dt$ is approximately the mass of type-$\boldsymbol{\theta}$ agents who switch to action i from other actions $j \in \mathcal{A}$, i.e., the gross **inflow** to $x_i(\boldsymbol{\theta})$; similarly, $x_i(\boldsymbol{\theta}) \sum_{j \in \mathcal{A}} R_{ij}(\boldsymbol{\pi}(\boldsymbol{\theta}))dt$ is the gross **outflow** from $x_i(\boldsymbol{\theta})$. Thus, $v_i(\boldsymbol{\pi}(\boldsymbol{\theta}), \mathbf{x}(\boldsymbol{\theta}))dt$ is the net flow to $x_i(\boldsymbol{\theta})$ in this period of time dt.

In a heterogeneous population game $\mathbf{F}$, the mean dynamic (4) of Bayesian strategy defines the **heterogeneous Bayesian dynamic** $\mathbf{v}^\mathbf{F}$ over $\mathcal{F}_\mathcal{X}$ by

$$\dot{\mathbf{x}}(\boldsymbol{\theta}) = \mathbf{v}^\mathbf{F}[\mathbf{x}](\boldsymbol{\theta}) := \mathbf{v}(\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta}), \mathbf{x}(\boldsymbol{\theta})) \in \mathbb{R}^A \quad \text{with } \bar{\mathbf{x}} = \mathbb{E}_\Theta \mathbf{x}$$

for each type $\boldsymbol{\theta} \in \Theta$.

Examples of evolutionary dynamics

To give a concrete image of switching rate functions, here we see major evolutionary dynamics.[10] In particular, we separate the dynamics based on optimization from others because they need different regularity conditions to guarantee existence of a unique solution trajectory.

Continuous dynamics. In the switching rate function of a continuous dynamic, the switching rate function R_{ij} continuously changes with the payoff vector.

Definition 1 (Continuous dynamics)**.** In a **continuous dynamic**, switching rate function R_{ij} : $\mathbb{R}^A \to \mathbb{R}_+$ of each pair of actions $i, j \in \mathcal{A}$ is Lipschitz continuous.

Example 3. In a class of **pairwise comparison target dynamics**, the switching rate R_{ij} depends only on the payoff difference $\pi_j(\boldsymbol{\theta}) - \pi_i(\boldsymbol{\theta})$. In particular, the switching rate function $R_{ij}(\boldsymbol{\pi}(\boldsymbol{\theta})) = [\pi_j(\boldsymbol{\theta}) - \pi_i(\boldsymbol{\theta})]_+$ defines the **Smith dynamic** (Smith, 1984).[11] ▮

Example 4. Because of continuity, we classify **smooth best response dynamics** (Fudenberg and Kreps, 1993) into continuous dynamics. For example, the **logit dynamic** (Fudenberg and Levine, 1998) is constructed from $R_{ij}(\boldsymbol{\pi}(\boldsymbol{\theta})) = \exp(\mu^{-1}\pi_j(\boldsymbol{\theta})) / \sum_{a\in\mathcal{A}} \exp(\mu^{-1}\pi_a(\boldsymbol{\theta}))$ with noise level $\mu > 0$. This switching rate function can be obtained from perturbed optimization: upon a receipt of a revision opportunity, an agent draws a random perturbation in each action a's payoff ε_a from the double exponential distribution[12] and then switches to the action that maximizes $\pi_a(\boldsymbol{\theta}) + \varepsilon_a$ among all actions $a \in \mathcal{A}$. In general, a smooth best response dynamic can be constructed from such perturbed optimization under some admissibility condition: see Hofbauer and Sandholm (2002, 2007). ▮

Note that payoff perturbation $\boldsymbol{\varepsilon} = (\varepsilon_a)_{a\in\mathcal{A}}$ is *transient* and a different value of $\boldsymbol{\varepsilon}$ will be drawn in the next revision opportunity, while the probability distribution is assumed to be i.i.d. So, there is no (ex ante) heterogeneity in $\boldsymbol{\varepsilon}$. In contrast, the idiosyncratic payoff vector $\boldsymbol{\theta}$ in our heterogeneous setting is *persistent*. To make comparison with heterogeneous dynamics, we can consider the **homogenized smooth BRD**, following the idea of Ely and Sandholm (2005): the unperturbed part of payoff vector $\boldsymbol{\pi}^0$ is common to all agents (i.e., $\boldsymbol{\theta}(\omega) \equiv \mathbf{0}$ for all $\omega \in \Omega$). Then, agents follow the smooth BRD with the transient payoff perturbation $\boldsymbol{\varepsilon}$ drawn from the distribution $\mathbb{P}_\Theta$.[13] Given $\boldsymbol{\pi}^0 = \mathbf{F}^0(\bar{\mathbf{x}})$, action a is the unique best response after drawing $\boldsymbol{\varepsilon}$ if $\boldsymbol{\varepsilon} \in \beta_a^{-1}[\bar{\mathbf{x}}]$. This happens with

[10] Readers who are familiar with major evolutionary dynamics may just scan this subsection quickly and jump to Definitions 1 and 2. Yet, it is recommended to check the homogenized smooth BRD and the tempered BRD, since they will play major roles in this paper.

[11] $[\cdot]_+$ is an operator to truncate the negative part of a number: i.e., $[\breve{\pi}]_+$ is $\breve{\pi}$ if $\breve{\pi} \geq 0$ and 0 otherwise.

[12] Given the noise level μ, the cumulative distribution function of the double exponential distribution is $\mathbb{P}(\varepsilon_a \leq c) = \exp(-\exp(-\mu^{-1}c - \gamma))$ where $\gamma \approx 0.5772$ is Euler's constant.

[13] To make this dynamic comparable with heterogeneous dynamics under *persistent* idiosyncratic payoffs $\boldsymbol{\theta} \in \mathbb{R}^A$, we reuse the distribution $\mathbb{P}_\Theta$ of $\boldsymbol{\theta}$ as the distribution of $\boldsymbol{\varepsilon} \in \mathbb{R}^A$. That is, an agent in the homogenized smooth BRD draws an idiosyncratic payoff vector from $\mathbb{P}_\Theta$ at *each* revision opportunity, while an agent in a heterogeneous dynamic draws an idiosyncratic payoff vector only once and keeps it forever.

probability $\mathbb{P}_\Theta(\beta_a^{-1}[\bar{\mathbf{x}}])$. Therefore, the homogenized smooth BRD is obtained as[14]

$$\dot{\bar{x}}_a = \mathbb{P}_\Theta(\beta_a^{-1}[\bar{\mathbf{x}}]) - \bar{x}_a \qquad \text{for each } a \in \mathcal{A}.$$

Note that, in a binary ASAG, this reduces to

$$\dot{\bar{x}}_I = P_\Theta(F_I^0(\bar{x}_I)) - \bar{x}_I,$$

since $\beta_I^{-1} = \{\boldsymbol{\theta} : \theta_O < F_I^0(\bar{x}_I)\}$. The sign of $\dot{\bar{x}}_I$ is thus identical with that of $F_I^0(\bar{x}_I) - P_\Theta^{-1}(\bar{x}_I)$.

Exact optimization dynamics. In an exact optimization dynamic, agents switches *only* to best responses to the current payoffs: if action j does not yield the maximal payoff among $\boldsymbol{\pi}(\boldsymbol{\theta}) = (\pi_1(\boldsymbol{\theta}), \ldots, \pi_A(\boldsymbol{\theta}))$, then $R_{ij}(\boldsymbol{\pi}(\boldsymbol{\theta})) = 0$ for any i. The switching rate to an optimal action can vary with $\boldsymbol{\pi}(\boldsymbol{\theta})$ and $i, j \in \mathcal{A}$. Denote by $Q_{ij}(\boldsymbol{\pi}(\boldsymbol{\theta}))$ the *conditional* switching rate from i to j *when* j is the unique best response. In the definition below, we extend the domain of Q_{ij} to $\mathbb{R}^A$ while assuming its continuity over the whole domain. The actual switching rate R_{ij} is defined as the truncation of Q_{ij} when j is not a best response; the truncation causes discontinuity.

Definition 2 (Exact optimization dynamics). In an **exact optimization dynamic**, switching rate function $R_{ij} : \mathbb{R}^A \to \mathbb{R}_+$ of each pair of actions $i, j \in \mathcal{A}$ is expressed as

$$R_{ij}(\boldsymbol{\pi}(\boldsymbol{\theta})) = \begin{cases} 0 & \text{if } j \notin \operatorname{argmax}_{a \in \mathcal{A}} \pi_a(\boldsymbol{\theta}), \\ Q_{ij}(\boldsymbol{\pi}(\boldsymbol{\theta})) & \text{if } \{j\} = \operatorname{argmax}_{a \in \mathcal{A}} \pi_a(\boldsymbol{\theta}), \end{cases}$$

with a Lipschitz continuous function $Q_{ij} : \mathbb{R}^A \to \mathbb{R}_+$.

Example 5. In the **standard BRD** (Hofbauer, 1995b; Gilboa and Matsui, 1991), a revising agent switches to the optimal action that maximizes the current payoff with sure. That is, the standard BRD is an exact optimization dynamic with $Q_{ij} \equiv 1$. The heterogeneous version is considered in Ely and Sandholm (2005); they prove that the aggregate strategy in the heterogeneous standard BRD follows the homogenized smooth BRD. ∎

Example 6. Consider a version of BRD in which the switching rate to the unique best response Q_{ij} depends on the payoff difference (the **payoff deficit**) between the current strategy i and the best response j, i.e., $Q_{ij}(\boldsymbol{\pi}(\boldsymbol{\theta})) = Q(\pi_j(\boldsymbol{\theta}) - \pi_i(\boldsymbol{\theta}))$ whenever $j \in \operatorname{argmax}_{a \in \mathcal{A}} \pi_a(\boldsymbol{\theta})$. Function $Q : \mathbb{R}_+ \to [0, 1]$ is called *tempering function* and assumed to be non-decreasing and, especially, strictly increasing in an interval $[0, \breve{\pi}^\sharp)$ with $\breve{\pi}^\sharp \in \mathbb{R}_+ \cup \{+\infty\}$ and continuously differentiable. Then this switching rate function yields the **tempered BRD** (Zusai, 2017b).[15] Given payoff function $\mathbf{F}$, we denote the payoff deficit of action i for type $\boldsymbol{\theta}$ at aggregate strategy $\bar{\mathbf{x}}$ by $\breve{F}_i[\bar{\mathbf{x}}](\boldsymbol{\theta}) := F_*[\bar{\mathbf{x}}](\boldsymbol{\theta}) - F_i[\bar{\mathbf{x}}](\boldsymbol{\theta})$. ∎

Note that, if there are only two actions, a continuous dynamic such as a pairwise comparison target dynamic reduces to an exact optimization dynamic as long as an agent never switches to a worse action than the current action, i.e., $R_{ij}(\boldsymbol{\pi}) = 0$ whenever $\pi_i > \pi_j$.

[14] Assumption 3 is assumed here to eliminate indeterminacy of the aggregate best response.

[15] In Zusai (2017b), this switching rate function can be further constructed from optimization with stochastic switching costs.

3 Existence of a unique solution trajectory

We verify the Lipschitz continuity of the dynamic to guarantee the existence of a unique solution trajectory. For this goal, we impose regularity assumptions. First of all, for simplicity to define the dynamic, we extend the domain of $\mathbf{F}(\boldsymbol{\theta})$ to $\mathbb{R}^A$: $\mathbf{F}(\boldsymbol{\theta})$ is a function that maps state $\bar{\mathbf{x}} \in \mathbb{R}^A$ to payoff vector $\boldsymbol{\pi} \in \mathbb{R}^A$. We impose the following regularity condition on the payoff function $\mathbf{F}$.

Assumption 1 (Regularity assumption on the payoff function)**.** Given $\boldsymbol{\theta} \in \Theta$, $\mathbf{F}(\boldsymbol{\theta}) : \mathbb{R}^A \to \mathbb{R}^A$ is Lipschitz continuous with Lipschitz constant $L_{\mathbf{F}}(\boldsymbol{\theta})$. In addition, $\bar{L}_{\mathbf{F}} := \mathbb{E}_\Theta L_{\mathbf{F}} < \infty$.

This assumption is satisfied in an ASAG, as long as the common payoff function $\mathbf{F}^0 : \mathbb{R}^A \to \mathbb{R}^A$ is Lipschitz continuous.

To ensure the existence of a unique solution trajectory, we assume that the switching rate is bounded. The assumption is satisfied in an ASAG, if the type distribution $\mathbb{P}_\Theta$ has a bounded support and the common payoff function $\mathbf{F}^0$ is continuous, even if the switching rate function itself is not bounded over the whole domain $\mathbb{R}^A$ like the Smith dynamic.

Assumption 2 (Bounded switching rates)**.** There exists $\bar{R} \in \mathbb{R}_+$ such that

$$R_{ij}(\mathbf{F}[\bar{\mathbf{m}}](\boldsymbol{\theta})) \leq \bar{R} \quad \text{with } \bar{\mathbf{m}} = \mathbb{E}_\Theta \mathbf{m}$$

for the density of any finite signed measure $\mathbf{m}$ and for any $i, j \in \mathcal{A}$ and $\boldsymbol{\theta} \in \Theta$.

For an exact optimization dynamic, Lipschitz continuity of Q_{ij} is not sufficient, as it does not guarantee Lipschitz continuity of switching rate function $\mathbf{R}$ due to truncation when the best response action changes. The switching rate R_{ij} is supposed to change only continuously thanks to continuity of Q_{ij} when action j remains to be the unique best response. However, payoff changes may cause change of the best response, which triggers discontinuous changes in the switching rate R_{ij} to the new best response from zero to some positive rate Q_{ij} and in the switching rate to the old one from positive to zero. The next assumption states that the mass of agents who experience switches of best responses due to payoff changes grows only continuously with the payoff change; thus, despite discontinuous changes in individual agents' switching rates, the sum of these changes over all the agents becomes continuous.

Assumption 3 (Continuous change in best response)**.** *If* the switching rate function $\mathbf{R} : \mathbb{R}^A \to \mathbb{R}_+^{A \times A}$ is an exact optimization protocol, then there exists $L_\beta \in \mathbb{R}_+$ such that

$$\mathbb{P}_\Theta(\beta_b^{-1}(\bar{\mathbf{x}}) \cap \beta_c^{-1}(\bar{\mathbf{x}}')) \leq L_\beta |\bar{\mathbf{x}} - \bar{\mathbf{x}}'|$$

for any two distinct actions $b, c \in \mathcal{A}$ such that $b \neq c$ and any two distinct states $\bar{\mathbf{x}}, \bar{\mathbf{x}}' \in \Delta^A$ such that $\bar{\mathbf{x}}' \neq \bar{\mathbf{x}}$.

Note that this assumption imposes a condition on the type distribution only for exact optimization dynamics; continuous dynamics do not need any such assumption on the type space for existence of a unique solution trajectory.

In an ASAG, Assumption 3 is satisfied if the distribution of differences in idiosyncratic payoffs between every two actions satisfies a Lipschitz-like continuity in the sense that there exists $\bar{p}_\Theta \in \mathbb{R}$ such that $\mathbb{P}_\Theta(\{\boldsymbol{\theta} \in \Theta : c \leq \theta_b - \theta_a \leq d\}) \leq (d-c)\bar{p}_\Theta$ for any $a, b \in \mathcal{A}$ and any $c, d \in \mathbb{R}$ such that $d > c$.

Theorem 1 (Existence of a unique solution trajectory under Bayesian dynamic)**.** *Consider a heterogeneous Bayesian dynamic $\mathbf{v}^{\mathbf{F}}$ in a population game $\mathbf{F}$ under a continuous dynamic or an exact optimization dynamic. Under Assumptions 1 to 3, the following holds.*

i) Function $\mathbf{v}^{\mathbf{F}}$ over $\mathcal{F}_\mathcal{X}$ is Lipschitz continuous in L^1-norm over $\mathcal{F}_\mathcal{X}$.

ii) There exists a unique solution trajectory $\{\mathbf{x}_t\}_{t\in\mathbb{R}_+} \subset \mathcal{F}_\mathcal{X}$ of $\dot{\mathbf{x}}_t = \mathbf{v}^{\mathbf{F}}[\mathbf{x}]$ from any initial Bayesian strategy $\mathbf{x}_0 \in \mathcal{F}_\mathcal{X}$.

4 Nonaggregability of heterogeneous dynamics

4.1 Generic nonaggregability of instantaneous transition

Given population game $\mathbf{F}$, the Bayesian dynamic $\dot{\mathbf{x}} = \mathbf{v}^{\mathbf{F}}[\mathbf{x}]$ is defined on the space of Bayesian strategy $\mathcal{F}_\mathcal{X}$: to predict the transition $\dot{\mathbf{x}}$, we need to identify the Bayesian strategy or equivalently the strategy composition over $\mathcal{A} \times \Theta$. Transition of the aggregate strategy $\dot{\bar{\mathbf{x}}}$ is obtained from aggregation of $\dot{\mathbf{x}}$: that is, $\dot{\bar{\mathbf{x}}}$ is obtained as

$$\dot{\bar{\mathbf{x}}} = \mathbb{E}_\Theta \dot{\mathbf{x}} = \mathbb{E}_\Theta \mathbf{v}^{\mathbf{F}}[\mathbf{x}].$$

As noted in the introduction, the preceding literature of heterogeneous evolutionary dynamics focused on aggregable dynamics, in which the transition of the aggregate strategy $\dot{\bar{\mathbf{x}}}$ can be identified by the current aggregate strategy $\bar{\mathbf{x}}$ alone. More specifically, we say that Bayesian dynamic $\mathbf{v}^{\mathbf{F}}$ over $\mathcal{F}_\mathcal{X}$ is **aggregable** if there is an aggregate dynamic $\bar{\mathbf{v}}^{\mathbf{F}} : \Delta^A \to \mathbb{R}^A$ such that $\{\mathbb{E}_\Theta \mathbf{x}_t\}_{t\in\mathbb{R}_+} \subset \Delta^A$ is a solution trajectory of $\bar{\mathbf{v}}^{\mathbf{F}}$ whenever $\{\mathbf{x}_t\}_{t\in\mathbb{R}_+} \subset \mathcal{F}_\mathcal{X}$ is a solution trajectory of $\mathbf{v}^{\mathbf{F}}$: that is,

$$\left[\, \bar{\mathbf{x}}_t = \mathbb{E}_\Theta \mathbf{x}_t \quad \text{and} \quad \dot{\mathbf{x}}_t = \mathbf{v}^{\mathbf{F}}[\mathbf{x}_t] \,\right] \qquad \Longrightarrow \qquad \dot{\bar{\mathbf{x}}}_t = \bar{\mathbf{v}}^{\mathbf{F}}(\bar{\mathbf{x}}_t)$$

for all $t \in \mathbb{R}_+$. As proven by Ely and Sandholm (2005, Theorem 5.4), this aggregability condition is equivalent to the interchangeability of aggregation and dynamic, or more specifically,

$$\mathbb{E}_\Theta \mathbf{v}^{\mathbf{F}}[\mathbf{x}] = \bar{\mathbf{v}}^{\mathbf{F}}\left(\mathbb{E}_\Theta \mathbf{x}\right) \qquad \text{for any } \mathbf{x} \in \mathcal{F}_\mathcal{X}.$$

They further verify that the standard BRD is aggregable and the aggregate strategy under the heterogeneous standard BRD follows the homogenized smooth BRD.

However, aggregability is indeed quite demanding for other evolutionary dynamics such as payoff comparison dynamics and imitative dynamics. The above condition requires the transition vector of an aggregate dynamic to vary only with the aggregate strategy, independently of the underlying strategy composition. An evolutionary dynamic is not aggregable when difference in the payoff vector over different types causes different switching rates. Nonaggregability is

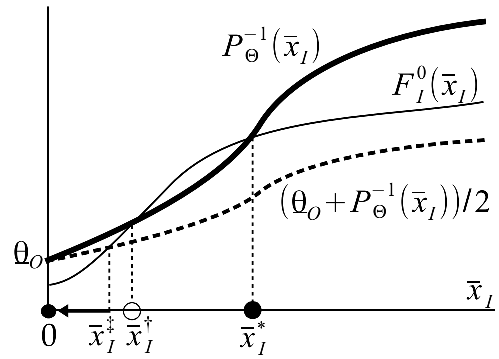


Figure 2: The common payoff function and the inverse c.d.f. of the type distribution in a binary coordination ASAG.

common to major evolutionary dynamics, since most of them—except the standard BRD—have the switching rate continuously changing with the payoff vector.

Theorem 2 (Generic nonaggregability)**.** *Consider a heterogeneous dynamic in an aggregate game with more than one payoff types. The dynamic is not aggregable and $\dot{\mathbf{x}}$ is not wholly determined from $\bar{\mathbf{x}}$ alone, unless $\bar{\mathbf{x}}$ is a pure strategy aggregate equilibrium or the variation in $R_{ji}(\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta})) + \sum_{k \neq i} R_{ik}(\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta}))$ is zero for every two distinct actions $i, j \in \mathcal{A}$ such that $i \neq j$ and $\bar{x}_j > 0$.*

In a binary aggregate game, the variation condition in the above theorem reduces to the zero variation in $R_{IO}(\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta})) + R_{OI}(\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta}))$, which we can call the unconditional total revision rate since it does not condition on the agent's current action.[16] In major evolutionary dynamics except smooth BRDs, an agent never switches to an action that is worse than the agent's current action. Thus, the unconditional total switching rate in a binary aggregate game is simply the switching rate from a suboptimal action to the optimal action. In a tBRD and any payoff comparison dynamic such as the Smith dynamic, the revision rate from a suboptimal action to the optimal action in a binary game is an increasing function of the payoff deficit $\check{F}_*[\bar{\mathbf{x}}](\boldsymbol{\theta})$. Thus, these dynamics are not aggregable. By the same token, we can confirm nonaggregability of excess payoff dynamics and imitative dynamics; see Section 6.

4.2 Nonaggregability of long-run outcome from a fixed initial state

The above nonaggregability theorem only states the condition under which the instantaneous transition of the aggregate state cannot be predicted from the current aggregate state. One may wonder if any such non-predictability can last even in the long run. Below we focus on a binary coordination ASAG and study the condition under which the aggregate state moves away from an aggregate equilibrium that is stable under the homogenized BRD when the difference in switching rates is large enough.

[16] Note that here we do not assume additive separability of payoff heterogeneity.

Consider a binary ASAG such as

$$F_I^0(\bar{x}_I) \begin{cases} > & \text{if } \bar{x}_I \in (\bar{x}_I^\dagger, \bar{x}_I^*), \\ = P_\Theta^{-1}(\bar{x}_I) & \text{if } \bar{x}_I \in \{\bar{x}_I^\dagger, \bar{x}_I^*\} \\ < & \text{if } \bar{x}_I \notin [\bar{x}_I^\dagger, \bar{x}_I^*], \end{cases}$$

where $0 < \bar{x}_I^\dagger < \bar{x}_I^* < 1$. Assume strict increasingness of F_I^0 (positive externality of action I) and continuous type distribution of θ_O, i.e., continuity of c.d.f. P_Θ, as well as Lipschitz continuity of F_I. Under the homogenized smooth BRD, $\bar{x}_I = 0$ and $\bar{x}_I = \bar{x}_I^*$ are stable aggregate equilibria and $\bar{x}_I = \bar{x}_I^\dagger$ is an unstable one. We call this game a **binary coordination ASAG.** Further, we assume that there is a lower bound $\underline{\theta}_O$ on the type space $\Theta_O \subset \mathbb{R}$.

Consider a heterogeneous dynamic starting from the initial Bayesian strategy $\mathbf{x}^0$ such as

$$x_I^0(\theta_O) = \begin{cases} 1 & \text{if } \theta_O > P_\Theta^{-1}(1 - \bar{x}_I^*), \\ 0 & \text{if } \theta_O < P_\Theta^{-1}(1 - \bar{x}_I^*). \end{cases}$$

In this composition, those who have relatively high values of the outside option θ_O happen to choose I while those who have lower values of the outside option happen to choose O; so their choices are initially opposite to their current best responses. We call this composition a *reversed composition*. The type $\hat{\theta}_O^0 := P_\Theta^{-1}(1 - \bar{x}_I^*)$ is the threshold between initial action-I players and O players. Assume $\hat{\theta}_O^0 > \theta_O^* := F_I^0(\bar{x}_I^*)$.

Yet, the aggregate strategy in this strategy composition coincides with aggregate equilibrium $\bar{x}_I^*$. Thus, the aggregate strategy must stay there under the homogenized smooth BRD; since it is also a stable equilibrium, it cannot leave this aggregate equilibrium even if there is so a small perturbation that keeps the aggregate strategy $\bar{x}_I$ above $\bar{x}_I^\dagger$.

However, the next theorem suggests that the aggregate strategy may escape from the "stable" aggregate equilibrium $\bar{x}_I^*$ and it may even converge to another aggregate equilibrium $\bar{x}_I = 0$. This depends on the difference in switching rates between those who switch from I to O and those who switch from O to I. The next theorem presents a rough sufficient condition that allows us to predict the escape just by comparing the switching rate of the threshold type $\hat{\theta}_O^0$ and that of the lowest type $\theta_O = \underline{\theta}_O$ at time 0.

Theorem 3 (Escape from a "stable" aggregate equilibrium in a binary coordination ASAG)**.** *Consider a heterogeneous dynamic in a binary coordination ASAG, starting from reversed composition $\mathbf{x}^0$ at time 0. Assume $\hat{\theta}_O^0 = P_\Theta^{-1}(\bar{x}_I^*) > \theta_O^* = F_I^0(\bar{x}_I^*)$. Suppose that the switching rate function R_{ij} is monotone to payoff gains from switches, in the sense that $R_{ij}(\boldsymbol{\pi})$ is an non-decreasing function of $\pi_j - \pi_i$ and $R_{ij}(\boldsymbol{\pi}) = 0$ only if $\pi_j \leq \pi_i$. Assume $r := R_{OI}(\mathbf{F}[\bar{\mathbf{x}}^*](\underline{\theta}_O))/R_{IO}(\mathbf{F}[\bar{\mathbf{x}}^*](\hat{\theta}_O^0)) < 1$. Then, the following holds.*

i) $\bar{x}_I$ decreases from $\bar{x}^*$ at least temporarily: $d\bar{x}_I/dt < 0$ at time 0.

ii) $\bar{x}_I$ reaches $\bar{x}_I^*(1 - (1 - r)r^{r/(1-r)})$ at some moment of time.

Furthermore, assume symmetry and strict payoff monotonicity of switching rate in the sense that there exists a strictly increasing function $R : \mathbb{R} \to \mathbb{R}_+$ such that $R_{IO}(\boldsymbol{\pi}) = R(\pi_O - \pi_I)$, $R_{OI}(\boldsymbol{\pi}) = R(\pi_I - \pi_O)$ and $R(\breve{\pi}) = 0$ for any $\breve{\pi} \leq 0$. Then, if there is an aggregate state $\bar{x}_I^{\ddagger}$ such that

$$F_I(\bar{x}_I) \leq 0.5 \left(P_{\Theta}^{-1}(\bar{x}_I) + \underline{\theta}_O \right) \qquad \textit{whenever } \bar{x}_I \leq \bar{x}_I^{\ddagger}$$

and the ratio of the initial switching rates r satisfies

$$\bar{x}_I^{*}(1 - (1-r)r^{r/(1-r)}) \leq \bar{x}_I^{\ddagger},$$

then the solution trajectory from the reversed composition $\mathbf{x}^0$ converges to $\bar{x}_I = 0$.

It is shown in the proof that the aggregate strategy $\bar{x}_I$ must decrease whenever $\bar{x}_I$ reaches $\bar{x}_I^{\ddagger}$, regardless of the underlying strategy composition; because of this, we call $\bar{x}_I^{\ddagger}$ a robust critical mass to decrease $\bar{x}_I$. The above inequality guarantees that it is surely reached in a finite time. After then, $\bar{x}_I$ must keep decreasing to $\bar{x}_I = 0$. Note that, $1 - (1-r)r^{r/(1-r)}$ is an increasing function of r, converging to 0 as $r \to 0$ and to 1 as $r \to 1$. Therefore, the condition is satisfied if r is sufficiently small, namely, if switching rates are so elastic to payoff differences and the difference in switching rates between different types is large enough.

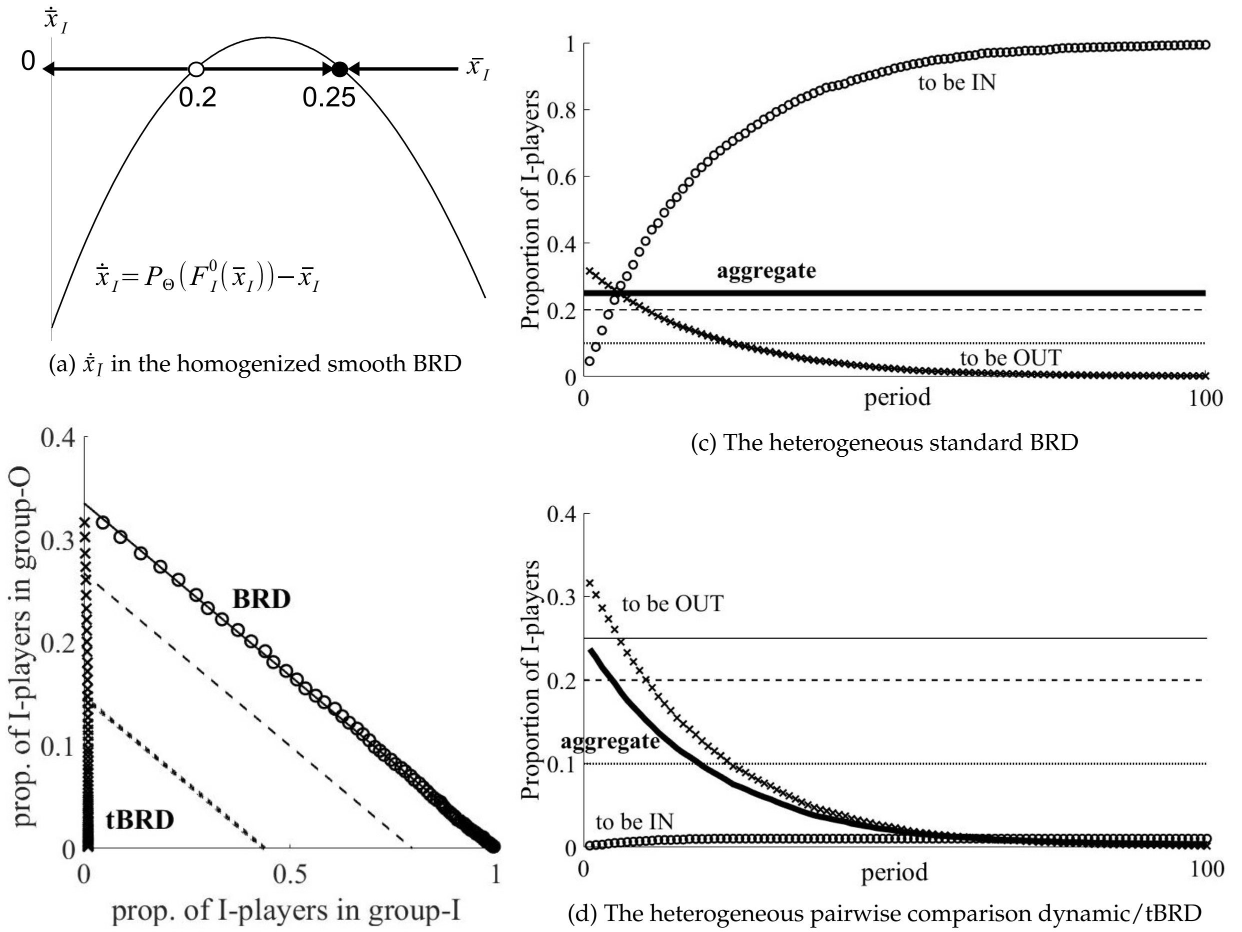
Note that the escape from the "stable" aggregate equilibrium $\bar{x}_I^{\ddagger}$ is not due to overshooting due to payoff perturbation or fluctuation in the initial aggregate strategy. The word "over" would infer so *strong* driving force *toward* an equilibrium that cannot be ceased even when the state reaches the equilibrium. However, as shown analytically in the proof of the above theorem and illustrated numerically in the next example, the aggregate dynamic starts *exactly* from $\bar{x}_I^{\ddagger}$, moves *away* from it since the very initial period, and then *monotonically* converges to another aggregate equilibrium $\bar{x}_I = 0$. In short, the cause of non-aggregability in the long-run outcome lies in the change in direction of the aggregate transition itself, not about the strength of the transition.

Example 7. Figure 3 shows the dynamic in such a binary coordination ASAG, in which the common payoff function is specified as $F_I^0(\bar{x}_I) = (49\bar{x}_I - 1)/20$ and the c.d.f. is $P_\Theta(\theta_O) = \sqrt{\theta_O + 1} - 1$ with support $\Theta_O = [0, 3]$. The interior stable equilibrium is $\bar{x}_I^* = 0.25$, and the initial aggregate strategy is set to this. While the unstable equilibrium is $\bar{x}_I^{\dagger} = 0.2$, $\bar{x}_I^{\ddagger} = 0.1$ is a robust critical mass.

Under the the standard BRD, the aggregate strategy remains at aggregate equilibrium $\bar{x}_I = 0.25$, as shown in Figure 3c. If we sort agents by θ_O, the lowest 25% of agents should take I and the others should take O in the Bayesian equilibrium with $\bar{x}_I = 0.25$. In these figures, the former group of agents is called *group I (to be IN)* and the latter group is called *group O (to be OUT)*. According to Figure 3b, the underlying strategy composition approaches the corresponding equilibrium composition, where the proportion of I-players in group I is 1 and that in group O is 0.

As an example of nonaggregable dynamics, we consider a pairwise comparison dynamic such as $R_{ij} = ([\pi_j - \pi_i]_+)^3$.[17] This nonaggregable dynamic drives the aggregate strategy away from $\bar{x}_I = 0.25$ and looks leading it to $\bar{x}_I = 0$ in Figure 3d. From a close look at this figure, we can see

[17]Due to binary actions, this can be also interpreted as a tempered BRD with $Q(\breve{\pi}) = \breve{\pi}^3$.



(a) $\dot{\bar{x}}_I$ in the homogenized smooth BRD

(b) Paths of composition

(c) The heterogeneous standard BRD

(d) The heterogeneous pairwise comparison dynamic/tBRD

Figure 3: Numerical simulations of the BRD and the pairwise comparison dynamic/tBRD in a binary coordination ASGA. In Figure 3b, the thin solid line shows the set of compositions that keep the aggregate strategy to one of aggregate equilibria $\bar{x}_I = 0.25$; the dashed line corresponds to another aggregate equilibrium $\bar{x}_I = 0.20$ and the dotted line to $\bar{x}_I = 0.1$. In Figures 3c and 3d, the horizontal lines show these aggregate equilibria as well.

that the switching agents in group I actually choose I in the first 100 periods, like in the standard BRD; but their switches are slower than the switches of agents in group O. So the aggregate share of *I*-players decreases.

The aggregate strategy leaves the aggregate equilibrium $\bar{x}_I = 0.25$ under the pairwise comparison dynamic, despite stability under the standard BRD, because those who switch from I to O face greater payoff gains than those from O to I and the pairwise comparison dynamic allows the former to switch faster than the latter. Thus, the outflow from I to O outweighs the inflow from O to I in the aggregate transition of $\bar{x}_I$ and thus lowers the aggregate proportion of I-players. In this example, this dominance continues so to push $\bar{x}_{I,t}$ down lower than the robust critical mass $\bar{x}_I = 0.10$. ∎

More detailed investigation of long-run nonaggregability in binary ASAGs is presented in a companion paper (Zusai, 2017a). The paper presents a general condition for robust critical masses and its application to equilibrium selection. It also exhibits generic non-stationarity of aggregate equilibrium under nonaggregable dynamics; in order to keep an aggregate strategy at an aggregate equilibrium, the distribution of the switching rates needs to satisfy some balancing condition, which is intuitively shown in the paper. Further, the paper shows that aggregate strategy generally fluctuates around an interior aggregate equilibrium, even if it is stable under a nonaggregable dynamic.

4.3 Implication on empirical methodology

There is a growing literature on estimation of discrete choice models in the presence of switching costs and heterogeneity over individuals. In health economics and IO/marketing research, it is possible to obtain micro data and thus directly observe the individual-level decision dynamic, i.e., the dynamic of strategic composition.[18] But, in many cases, researchers may not have an access to desirable micro data. To study dynamics at individual level, one needs to track the behavior of each individual over time and thus needs to identify each individual in the sample and tag the identity with the data. Such well-designed micro data may not be collected if there is no particular intention for the data collecting agency to analyze the panel data. For example, a local transportation agency may be monitoring the aggregate traffic volumes on each major street but may not track each individual car's trip. Even if there is such data, privacy protection may be so strict or data collecting companies may charge so high price on access to data that micro panel data is not accessible to academic researchers.

Hence, applied economists may hope to do some rough study on aggregate dynamic from aggregate data. From theoretical analysis of Bertrand competition with consumers subject to switching costs, Shy (2002) proposes a formula to calculate the switching costs from product prices and the aggregate strategy distribution by utilizing the equilibrium condition on these aggregate variables and this method is widely used in applied empirical research. But, as Shy noted, his model assumes homogeneity of agents except their initial choices and also presumes equilibrium.[19] In an empirical study on international migration, Artuç, Lederman, and Porto (2015) acknowledge the limitation of such reduced-form approach: they run Monte Carlo simulation using micro data on labor mobility in the United States. They find that, if heterogeneity is persistent and also individual decisions incur switching costs, the logit estimation based on the aggregate data yields sample selection biases in the degree of estimated parameters compared to the regression directly from the micro-data. The main issue of their study is international migrations from developing countries and thus they do not have micro data on this issue. Hence, they have to admit that only transient heterogeneity as assumed in the homogenized smooth BRD is captured in their econometric framework.

[18] For example, see Goettler and Clay (2011) and Handel (2013).

[19] Shy admits that equilibrium does not exist in the Bertrand game, though he makes justification of the use of the equilibrium condition (a variant of the condition for no profitable deviation). Also, Shy's paper belongs to applications of economic theory and is not intended to offer a rigorous econometric theory, unlike the current most advanced empirical study in estimations of switching costs.

We share the same concern on over-simplification of aggregate dynamic in the presence of heterogeneity and indeed put it further. The theoretical study in this paper suggests that, even if an economist somehow *precisely knows* the common payoff function, the distribution of idiosyncratic payoffs, and the switching rate functions of individuals, an aggregable dynamic that assumes transitory payoff heterogeneity yields a different prediction on the change in aggregate strategy compared to the nonaggregable dynamic in which payoff heterogeneity is persistent. The difference is not only in magnitudes of changes but also in directions of changes and indeed in long-run outcomes.

5 Bayesian equilibrium: stationarity and stability

5.1 Extension of stationarity and stability to heterogeneous dynamics

Our dynamic could be seen as extension of evolutionary dynamics in a single homogeneous population to continuously many heterogeneous populations, though the existence of a unique solution trajectory requires careful formulation of the state space. despite negative results of generic nonaggregability, it is natural to expect that stationarity and stability of Nash equilibrium are extended to *Bayesian equilibrium* in the heterogeneous setting.

As long as agents obey the same switching rate function $\mathbf{R} = (R_{ij})_{i,j\in\mathcal{A}} : \mathbb{R}^A \to \mathbb{R}_+^{A\times A}$, the homogeneous mean dynamic of the action distribution $\mathbf{x}^0 \in \Delta^A$ over all the agents in the society follows the same function $\mathbf{v} : \Delta^A \times \mathbb{R}^A \to \mathbb{R}^A$ as the heterogeneous mean dynamic (4) of the action distribution $\mathbf{x}(\boldsymbol{\theta}) \in \Delta^A$ over all the agents with the same type $\boldsymbol{\theta}$; each individual agent's switch from current action i to new action j is completely determined from the agent's payoff vector through the switching rate function $R_{ij} : \mathbb{R}^A \to \mathbb{R}_+$ and those switches over the agents who face the same payoff vector are simply summed to the mean dynamic $\mathbf{v}$. The difference lies only in what payoff vector to be plugged in $\mathbf{v}$.

More specifically, in the homogeneous setting, the payoff vector $\boldsymbol{\pi}^0 \in \mathbb{R}^A$ is common to all the agents in the society and the action distribution is just A-dimensional, i.e., $\mathbf{x}^0 \in \Delta^A$. Thus, the homogeneous version of each evolutionary dynamic is straightforwardly obtained by plugging $\boldsymbol{\pi}^0$ into the switching rate function $\mathbf{R}$ of the dynamic. That is, the homogeneous mean dynamic is obtained as

$$\dot{x}_i^0 = \sum_{j\in\mathcal{A}} x_j^0 R_{ji}(\boldsymbol{\pi}^0) - x_i^0 \sum_{j\in\mathcal{A}} R_{ij}(\boldsymbol{\pi}^0) \qquad \text{for all } i \in \mathcal{A}. \tag{5}$$

Compare this equation with (4) that defines the mean dynamic function $\mathbf{v}$ and then we can find $\dot{x}_i^0 = v_i(\boldsymbol{\pi}^0, \mathbf{x}^0)$ for each i and thus $\dot{\mathbf{x}}^0 = \mathbf{v}(\boldsymbol{\pi}^0, \mathbf{x}^0)$. In homogeneous population game $\mathbf{F}^0 : \Delta^A \to \mathbb{R}^A$, this induces the homogeneous evolutionary dynamic $\mathbf{v}^{\mathbf{F}^0}$ over Δ^A such as

$$\dot{\mathbf{x}}^0 = \mathbf{v}^{\mathbf{F}^0}(\mathbf{x}^0) := \mathbf{v}(\mathbf{F}^0(\mathbf{x}^0), \mathbf{x}^0).$$

Stationarity of Bayesian equilibrium

To link the homogeneous dynamic $\dot{\mathbf{x}}^0 = \mathbf{v}^{\mathbf{F}^0}(\mathbf{x}^0)$ on Δ^A and the heterogeneous Bayesian dynamic $\dot{\mathbf{x}} = \mathbf{v}^{\mathbf{F}}[\mathbf{x}]$ on $\mathcal{F}_{\mathcal{X}}$, we first identify the properties of the mean dynamic $\mathbf{v}$ that induce stationarity and stability of equilibrium, separately from the population game. This separation is useful because both homogeneous and heterogeneous dynamics stem from the same mean dynamic $\mathbf{v}$ (constructed from the same switching rate function $\mathbf{R}$). Their difference is found in the population game played by agents, namely $\mathbf{F}$ or $\mathbf{F}^0$.

In the homogeneous setting, stationarity of Nash equilibrium is an immediate consequence of stationarity of the action distribution in which agents are taking the best response to the current payoffs, or shortly *best response stationarity*: the action distribution does not change if every agent is taking an optimal action given the current payoff vector.

Definition 3 (Best response stationarity of mean dynamic)**.** Mean dynamic $\mathbf{v} : \Delta^A \times \mathbb{R}^A \to \mathbb{R}^A$ satisfies **best response stationarity** if, for any $\boldsymbol{\pi}^0 \in \mathbb{R}^A, \mathbf{x}^0 \in \Delta^A$,

$$\mathbf{v}(\boldsymbol{\pi}^0, \mathbf{x}^0) = \mathbf{0} \qquad \iff \quad \forall b \in \mathcal{A}[x_b^0 > 0 \quad \Rightarrow \quad \pi_b^0 \geq \pi_a^0 \ \forall a \in \mathcal{A}]. \tag{6}$$

All the dynamics mentioned in Section 2.3, except smooth BRDs, satisfy the best response stationarity.[20] The logit dynamic satisfies a version of these properties, modified for logit choice; similar for other smooth BRDs. In a homogeneous population game, best response stationarity implies stationarity of Nash equilibrium and non-stationarity of non-equilibrium states:

$$\mathbf{v}^{\mathbf{F}^0}(\mathbf{x}^0) = \mathbf{0} \quad \iff \quad \mathbf{x}^0 \text{ is a Nash equilibrium in } \mathbf{F}^0.$$

In the heterogeneous setting, best response stationarity applies to each type: the action distribution of a particular type $\boldsymbol{\theta}$ remains unchanged if and only if almost all agents of this type choose the best response to the current payoff for this type. Thus, it is natural that best response stationarity implies stationarity of a Bayesian equilibrium and non-stationarity of non-equilibrium Bayesian strategies.

Theorem 4 (Stationarity of Bayesian equilibrium)**.** *Suppose that mean dynamic* $\mathbf{v}$ *satisfies the best response stationarity* (6)*. Then, in any heterogeneous population game* $\mathbf{F}$*, a Bayesian equilibrium is stationary under the heterogeneous evolutionary dynamic* $\mathbf{v}^{\mathbf{F}}$*:*

$$\mathbf{v}^{\mathbf{F}}[\mathbf{x}](\boldsymbol{\theta}) = \mathbf{0} \text{ for } \mathbb{P}_\Theta\text{-almost all } \boldsymbol{\theta} \in \Theta \quad \iff \quad \mathbf{x} \text{ is a Bayesian equilibrium in } \mathbf{F}. \tag{7}$$

We saw that stationarity of aggregate equilibrium is not guaranteed any more. But Theorem 4 mitigates concerns about equilibrium stationarity. If the underlying Bayesian strategy is exactly a Bayesian equilibrium and thus is perfectly sorted, it stays there; as a consequence, the aggregate strategy also remains at the corresponding aggregate equilibrium. This clarifies that the driving

[20]In the homogeneous version of exact optimization dynamics, the best response stationarity needs to further assume $R_{ij}(\boldsymbol{\pi}) = 0$ when the current action i is a best response to $\boldsymbol{\pi}$; this was not assumed in our definition in cases of multiple best responses. In the heterogeneous setting, this concern on multiple best responses is eliminated by the assumption of non-atomic type distribution. That is, Assumption 3 replaces the assumption of $R_{ij}(\boldsymbol{\pi}) = 0$ for best response i to $\boldsymbol{\pi}$.

force to move the aggregate strategy from an aggregate equilibrium is indeed the nonaggregable sorting pressure on the underlying strategy composition toward a perfectly sorted composition.

Stability of Bayesian equilibrium in potential games

While stability of Nash equilibrium is not generally guaranteed even in the homogeneous setting, it is verified for potential games under a wide class of evolutionary dynamics. For a game played in large population, a potential game is defined as a game whose payoff vector can be derived as the derivative of some scalar-valued function, i.e., a potential function. It is equivalent to externality symmetry: the change in the payoff of an action by change in the mass of another action's players is symmetric between these two actions. The class of potential games includes random matching in symmetric games, binary games and congestion games. Sandholm (2010, Chapter 3) provides further explanation and examples.

Definition 4 (Potential game in the homogeneous setting)**.** Homogeneous population game $\mathbf{F}^0 : \Delta^A \to \mathbb{R}^A$ is called a **potential game** if there is a scalar-valued continuously differentiable function $f^0 : \mathbb{R}^A \to \mathbb{R}$ whose gradient vector always coincides with the payoff vector: for all $\bar{\mathbf{x}} \in \Delta^A$, f^0 satisfies

$$\frac{\partial f^0}{\partial \bar{x}_a}(\bar{\mathbf{x}}) = F^0_a(\bar{\mathbf{x}}) \text{ for all } a \in \mathcal{A},$$

$$\text{i.e., } \nabla f^0(\bar{\mathbf{x}}) := \left(\frac{\partial f^0}{\partial \bar{x}_1}(\bar{\mathbf{x}}), \ldots, \frac{\partial f^0}{\partial \bar{x}_A}(\bar{\mathbf{x}})\right) = \mathbf{F}^0(\bar{\mathbf{x}}).$$

Definition 5 (Potential game in the heterogeneous setting)**.** Aggregate heterogeneous population game $\mathbf{F} : \Delta^A \to \mathcal{C}_\Theta$ is called an (aggregate) **potential game** if there is a scalar-valued Fréchet-differentiable function $f : \mathcal{X} \to \mathbb{R}$ that is continuous in the weak topology on the strategy composition space $\mathcal{X}$ and whose Fréchet-derivative at each composition $\mathbf{X} \in \mathcal{X}$ coincides with $\mathbf{F}[\bar{\mathbf{x}}]$ at the corresponding aggregate state $\bar{\mathbf{x}} \in \Delta^A$.[21]

Both in the homogeneous and heterogeneous settings, all local maxima and interior local minima of a potential function, and indeed all the solutions of the Karash-Kuhn-Tucker first-order conditions for extrema are equilibria in a potential game; see Sandholm (2001) for the proof for Nash equilibrium in a homogeneous potential game and Sandholm (2005, Appendix A.3) for Bayesian equilibrium in a heterogeneous potential game.

While the potential f is defined as a function of strategy composition $\mathbf{X} \in \mathcal{X}$, we can say that the potential of a Bayesian strategy $\mathbf{x} \in \mathcal{F}_\mathcal{X}$ is $f(\int \mathbf{x} d\mathbb{P}_\Theta)$ where $\int \mathbf{x} d\mathbb{P}_\Theta \in \mathcal{X}$ is the corresponding strategy composition. Below we abuse the notation of f to mean $f(\int \mathbf{x} d\mathbb{P}_\Theta)$ by $f(\mathbf{x})$, justified by one-to-one correspondence between $\mathbf{X}$ and $\mathbf{x}$, as discussed in Appendix A.1.

[21] That is, at each strategy composition $\mathbf{X} \in \mathcal{X}$ with the aggregate strategy $\bar{\mathbf{x}} = \mathbf{X}(\Theta)$, the payoff vector function $\mathbf{F}[\bar{\mathbf{x}}] \in \mathcal{C}_\Theta$ satisfies $f(\mathbf{X}') = f(\mathbf{X}) + \langle \mathbf{F}[\bar{\mathbf{x}}], \mathbf{X}' - \mathbf{X} \rangle + o(\|\mathbf{X}' - \mathbf{X}\|^\infty_{\mathcal{A}\times\Theta})$ for any $\mathbf{X}' \in \mathcal{X}$. Here, operator $\langle \cdot, \cdot \rangle$ is defined as $\langle \boldsymbol{\pi}, \Delta\mathbf{X} \rangle = \int_\Theta \sum_{a \in \mathcal{A}} \pi_a(\boldsymbol{\theta}) \Delta X_a(d\boldsymbol{\theta}) = \mathbb{E}_\Theta[\boldsymbol{\pi}(\boldsymbol{\theta}) \cdot \Delta\mathbf{x}(\boldsymbol{\theta})]$ for each $\boldsymbol{\pi} \in \mathcal{C}_\Theta$ and $\Delta\mathbf{X} = \int \Delta\mathbf{x} d\mathbb{P}_\Theta \in \mathcal{M}_{\mathcal{A}\times\Theta}$. The norm $\|\cdot\|^\infty_{\mathcal{A}\times\Theta}$ is the variational norm on $\mathcal{X}$ to metrize the strong topology: we have $\|\Delta\mathbf{X}\|^\infty_{\mathcal{A}\times\Theta} = \sum_{a \in \mathcal{A}} \mathbb{E}_\Theta |\Delta x_a|$ by Theorem 8 in Appendix A.2. Fréchet-differentiability is defined for the strong topology and thus continuity in the weak topology is additionally required.

The key property of evolutionary dynamics for equilibrium stability in potential games is the *positive correlation*: net increase of each action's player and the action's payoff is positively correlated and the correlation is strictly positive unless the strategy distribution is unchanged. Major evolutionary dynamics, except smooth BRDs, satisfy the positive correlation.[22]

Definition 6 (Positive correlation of mean dynamic)**.** Mean dynamic $\mathbf{v} : \Delta^A \times \mathbb{R}^A \to \mathbb{R}^A$ satisfies the **positive correlation** if, for any $\boldsymbol{\pi}^0 \in \mathbb{R}^A, \mathbf{x}^0 \in \Delta^A$,

$$\boldsymbol{\pi}^0 \cdot \mathbf{v}(\boldsymbol{\pi}^0, \mathbf{x}^0) \begin{cases} \geq 0 & \text{for any } \boldsymbol{\pi}^0 \in \mathbb{R}^A, \mathbf{x}^0 \in \Delta^A; \\ > 0 & \text{if } \mathbf{v}(\boldsymbol{\pi}^0, \mathbf{x}^0) \neq \mathbf{0}. \end{cases} \tag{8}$$

In a homogeneous potential game, the positive correlation immediately implies that the homogeneous potential function increases over time until it reaches a stationary point, which is indeed a Nash equilibrium by equilibrium stationarity, since the definition of the potential function implies

$$\frac{d}{dt} f^0(\mathbf{x}^0) = Df^0(\mathbf{x}^0)\dot{\mathbf{x}}^0 = \mathbf{F}^0(\mathbf{x}^0) \cdot \dot{\mathbf{x}}^0 \begin{cases} \geq 0 & \text{for any } \mathbf{x}^0 \in \Delta^A; \\ > 0 & \text{if } \mathbf{v}^{\mathbf{F}^0}(\mathbf{x}^0) \neq \mathbf{0}. \end{cases}$$

Thus, the homogeneous *potential* function f^0 works as a Lyapunov function commonly in these evolutionary dynamics. Therefore, the positive correlation guarantees that the set of local maxima of f^0 is globally attracting and a strict local maximum is asymptotically stable (Sandholm, 2001). As a local maximum of the potential function is a Nash equilibrium, this implies global convergence to the set of Nash equilibria.

In the heterogeneous setting, the positive correlation applies to positive correlation between the payoffs and the action distribution among *each type*'s agents. Thus, by the same token as in a homogeneous potential game, this implies in a heterogeneous potential game $\mathbf{F}$ that the heterogeneous potential function f works as a Lyapunov function in order to extend equilibrium stability. Again, since a local maximum of f is a Bayesian equilibrium, the following theorem suggests global convergence to the set of Bayesian equilibria, which include all the local maxima of f.

Theorem 5 (Stability of Bayesian equilibria in aggregate potential games)**.** *Suppose that mean dynamic* $\mathbf{v}$ *satisfies the best response stationarity* (6) *and the positive correlation* (8)*. Then, in an aggregate potential game* $\mathbf{F}$*, the following holds.*

i) The set of local maxima of f is globally attracting under $\mathbf{v}^{\mathbf{F}}$; a local strict maximum of f is locally asymptotically stable.

ii) Let $\mathbf{x}^*$ be an isolated aggregate equilibrium in the sense that, in a neighborhood O^* of the corresponding strategy composition $\mathbf{X}^*$ in the composition space $\mathcal{X}$, there is no other equilibrium composition than $\mathbf{X}^*$. If $\mathbf{x}^*$ is (locally) asymptotically stable, then it is a local strict maximum of f.

If a dynamic satisfies best response stationarity and the positive correlation, call it an **admissible** dynamic.

[22] See Sandholm (2010, Chapter 5) for summary of the relationship between dynamics and the two properties in this section.

Corollary 1. *Assume Assumptions 1 to 3. Pairwise comparison target dynamics and exact optimization dynamics are admissible dynamics.*[23] *Therefore, a Bayesian equilibrium and a stationarity Bayesian strategy are equivalent under these dynamics in any heterogeneous population games; besides, stability of Bayesian equilibrium holds for these dynamics in heterogeneous potential games.*

Combined with aggregability of the standrd BRD verified by Ely and Sandholm (2005), Theorem 5 implies equivalence of local stability between (any) admissible dynamics—both aggregable and nonaggregable—and the homogenized smooth BRD. In a potential game, once an (isolated) Bayesian equilibrium is found to be locally stable in the heterogeneous standard BRD—or equivalently, if the corresponding aggregate equilibrium is locally stable in the homogenized smooth BRD, then its local stability is maintained in any admissible heterogeneous dynamics. Therefore, despite nonaggregability in general, we can test local stability of a Bayesian equilibrium under an arbitrary admissible heterogeneous dynamic just by examining the homogenized smooth BRD, as long as we know that the aggregate game is a potential game.

Corollary 2 (General aggregability of local stability)**.** *Consider aggregate potential game* $\mathbf{F}$ *with type distribution* $\mathbb{P}_\Theta$*.*

Let $\bar{\mathbf{x}}^* \in \Delta^A$ *be an isolated aggregate equilibrium in the sense that, in a neighborhood* $\bar{O}^*$ *of* $\bar{\mathbf{x}}^*$ *in the aggregate strategy space* Δ^A*, there is no other aggregate equilibrium than* $\bar{\mathbf{x}}^*$*. Correspondingly, let* $\mathbf{x}^* \in \mathcal{F}_\mathcal{X}$ *be the Bayesian equilibrium such that* $\mathbb{E}_\Theta \mathbf{x}^* = \bar{\mathbf{x}}^*$*. Then, the following statements are equivalent:*

i) $\bar{\mathbf{x}}^*$ *is locally asymptotically stable under the homogenized smooth BRD.*

ii) $\mathbf{x}^*$ *is a local strict maximum of the heterogeneous potential* f*.*

iii) $\mathbf{x}^*$ *is locally asymptotically stable under a heterogeneous admissible dynamic.*[24]

Example 8. Consider a (reduced) binary game $\mathcal{A} = \{I, O\}$ with *negative* externality: $F_I^0(\bar{x}_I)$ decreases with $\bar{x}_I \in [0,1]$. Then, the potential function f^0 is strictly concave; due to the bounded state space $\bar{x}_I \in [0,1]$ and strict concavity of f^0, the global maximum exists uniquely and there is no other local maximum of f^0. The global maximum of f^0 is the only aggregate equilibrium of this game. According to Theorem 2, this aggregate equilibrium is globally asymptotically stable under any admissible heterogeneous dynamics.

For an example in microeconomic theory to fall into this class of games, consider an entry-exit dynamic of producers in an industry. To make entry and exit symmetric, it is conventionally assumed that fixed costs exist but they are not sunk: fixed costs are paid only to maintain production capacities and they are revocable when the supplier becomes inactive. Further, the choice

[23]Observational dynamics such as the replicator dynamic (if $\mathbf{x}(\boldsymbol{\theta}) \gg \mathbf{0}$ for almost all $\boldsymbol{\theta}$) and excess payoff dynamics can be also included in Corollary 1.

[24]It is sufficient for the other conditions if it is locally asymptotically stable under *some* admissible heterogeneous dynamic, while each of the other conditions implies locally asymptotically stability under *any* admissible heterogeneous dynamics.

of entry and exit is regarded as a "long-term" decision while the choice of quantity supplied is a "short-term" decision (as well as the underlying consumers' decisions on demands); thus, it is commonly assumed that the market is settled to market equilibrium (demand equals to supply) at each moment of time, given the mass (number) of active suppliers at the moment. A free-entry or so-called "long run" equilibrium is defined in the homogeneous setting as a state in which gross profit for an active producer is equal to the fixed entry cost.

One may want to introduce heterogeneity to the fixed costs; it not only sounds realistic but also eliminates indeterminacy of individual choices at a free-entry equilibrium. Under heterogeneity in fixed costs, a free-entry equilibrim should be redefined as a state in which all the active producers have smaller fixed costs than the gross profit and all the inactive ones have greater fixed costs.

Under perfect competition in a standard setting as in Mas-Colell, Whinston, and Green (1995, Section 10.F), instantaneous market-equilibrium profit of an active supplier decreases with the number of active suppliers. We can regard $F^0_I(\bar{x}_I)$ as gross profit at this instantaneous competitive equilibrium given the current mass $\bar{x}_I$ of active suppliers and $\theta_O(\omega)$ as the fixed costs of supplier ω; then, the choice between entry and exit in perfect competition falls into the binary game with negative externality.

Thanks to our stability result, we can justify a free-entry equilibrium as the globally stable state in an evolutionary dynamic; indeed it is so strengthened to be stable in any admissible dynamics. As argued in Zusai (2017b), the tBRD is considered as a version of the BRD in which a revising agent pays a stochastic switching cost. Thus, the stability in the tBRD implies that, even if entry and exit incur sunk costs to build or scrap the production capacity, the "long-run" equilibrium is indeed the limit state under such an entry-exit dynamic.

By the same token, we can justify a free-entry equilibrium in the standard static monopolistic competition model such as Dixit and Stiglitz (1977) as a dynamically stable state and under an arbitrary admissible dynamic. ∎

An application to dynamic implementation of the social optimum

Imagine a central planner whose goal is to maximize the total payoff

$$\mathbb{E}_\Theta\left[\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta}) \cdot \mathbf{x}(\boldsymbol{\theta})\right] = \mathbf{F}^0(\bar{\mathbf{x}}) \cdot \bar{\mathbf{x}} + \mathbb{E}_\Theta\left[\boldsymbol{\theta} \cdot \mathbf{x}(\boldsymbol{\theta})\right] \quad \text{with } \bar{\mathbf{x}} = \mathbb{E}_\Theta \mathbf{x}.$$

To help the central planner achieve this goal, we introduce monetary payment to the agent's payoff: now a type-$\boldsymbol{\theta}$ agent's payoff from action $i \in \mathcal{A}$ is $\tilde{F}^{\mathbf{T}}_i[\mathbf{x}](\boldsymbol{\theta}) := F_i[\mathbf{x}](\boldsymbol{\theta}) - T_i[\bar{\mathbf{x}}]$, where $\mathbf{T} = (T_i)_{i\in\mathcal{A}} : \Delta^A \to \mathbb{R}^A$ is a function to determine the monetary transfer (in terms of payoff) from the agent to the planner for taking each action at state $\mathbf{x} \in \mathcal{F}_\mathcal{X}$. Sandholm (2002) proposes the dynamic Pigouvian pricing such as

$$T_i[\bar{\mathbf{x}}] = -\sum_{j\in\mathcal{A}} \bar{x}_j \frac{\partial F^0_j}{\partial \bar{x}_i}(\bar{\mathbf{x}}) \quad \text{for each } \bar{\mathbf{x}} \in \Delta^A.$$

Notice that this pricing scheme does not require the central planner to know the agents' switching rate functions, the type distribution, or even the current strategy composition.

Strictly speaking, in a setting where there are *finitely many* payoff types, Sandholm (2002) verified that, with $\mathbf{T}$ being the above dynamic Pigovian pricing scheme, $\tilde{\mathbf{F}}^{\mathbf{T}}$ has a potential function $\tilde{f}^{\mathbf{T}}$ being the sum of total payoffs over the society:

$$\tilde{f}^{\mathbf{T}}(\mathbf{x}) = \mathbb{E}_{\Theta}\left[\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta}) \cdot \mathbf{x}(\boldsymbol{\theta})\right] \quad \text{with } \bar{\mathbf{x}} = \mathbf{X}(\Theta).$$

In particular, if common payoff function $\mathbf{F}^0$ exhibits negative externality, $\tilde{f}^{\mathbf{T}}$ is concave and thus the unique social optimum is achieved in the long run through this pricing scheme regardless of the initial state. Thanks to Theorem 5 and corollary 2, now we can extend this claim to the games with *infinitely many* payoff types.[25]

5.2 Comparison with homogenized smooth BRD in potential ASAGs

Theorem 5 implies that an aggregate strategy must converge to the set of aggregate equilibria in a heterogeneous potential game even in nonaggregable dynamics. Recall that the binary game in Example 7 is indeed a potential game, and we witnessed that aggregate strategy moves from one aggregate equilibrium to another. Perhaps, one might feel hard to accord the positive result in the above theorem and the negative result in this example. The key to understand this gap is topological difference between Bayesian strategy and aggregate strategy. Here we highlight this gap by assuming additive separability of payoff heterogeneity, which enables us to construct the homogenized version of a potential function.

Extension of a homogeneous potential game to an ASAG

We extend a homogeneous potential game $\mathbf{F}^0$ to a heterogeneous aggregate game by keeping $\mathbf{F}^0$ as a common payoff function and introducing additively separable idiosyncratic payoffs $\boldsymbol{\theta}$ just as defined in (1). Then, this heterogeneous game is an aggregate potential game; we call it a potential ASAG. The payoff perturbation adds a nonaggregable term to the potential function and indeed makes its value vary with Bayesian strategy. Note that the additive separability of idiosyncratic payoffs and continuity of f^0 imply Assumption 1.

Theorem 6 (Extension of a homogenous potential game to an ASAG)**.** *Consider the common payoff function $\mathbf{F}^0$ that admits a homogeneous potential function $f^0 : \Delta^A \to \mathbb{R}$ such that $\nabla f^0 \equiv \mathbf{F}^0$, and the heterogeneous population game $\mathbf{F}$ derived from $\mathbf{F}^0$ by* (1)*. Then, $\mathbf{F}$ is a potential game with a **heterogeneous potential function** f given by*[26]

$$f(\mathbf{x}) = f^0(\mathbb{E}_{\Theta}\mathbf{x}) + \mathbb{E}_{\Theta}[\boldsymbol{\theta} \cdot \mathbf{x}(\boldsymbol{\theta})]. \tag{9}$$

[25] Sandholm (2005, p.903) speculated it by referring to Ely and Sandholm (2005), which allows us to reduce stability in the heterogeneous standard BRD to the one in the homogenized smooth BRD. However, we have found that aggregation is not an valid approach to other heterogeneous dynamics including the Smith dynamic, which is first proposed and popularly used in transportation engineering.

[26] This function f appears in the study of evolutionary implementation by Sandholm (2005, Appendix A.3). But it is only to characterize a Bayesian equilibrium as a solution of the KKT condition for local maxima and minima of f.

Difference in topology and basin of attraction

A Lyapunov function allows us to summarize the dynamic on a multi-dimensional space into a one-dimensional dynamic of the value of this scalar-valued function. In a potential game, the potential function serves as a Lyapunov function. As we saw in Example 7, a heterogeneous dynamic may behave differently from the homogenized smooth BRD and may even converge to a different equilibrium, despite equivalence in local stable equilibria in Corollary 2. The difference between these dynamics is clarified by looking at the difference in the Lyapunov functions that represent these dynamics.

Hofbauer and Sandholm (2007) prove that, under the homogenized smooth BRD, the Lyapunov function $\bar{f} : \mathcal{X} \to \mathbb{R}$ in a potential game is constructed as[27]

$$\bar{f}(\bar{\mathbf{x}}) := f^0(\bar{\mathbf{x}}) + \min_{\bar{\boldsymbol{\pi}} \in \mathbb{R}^A} \left(\mathbb{E}_\Theta[\max_{a \in \mathcal{A}}(\bar{\pi}_a + \theta_a)] - \bar{\boldsymbol{\pi}} \cdot \bar{\mathbf{x}}. \right) \tag{10}$$

from the potential function $f^0 : \Delta^A \to \mathbb{R}$. We call $\tilde{f}$ the **homogenized potential function**, to compare it with the heterogeneous potential function f. Actually, the next theorem states close connection between f and $\bar{f}$.

Theorem 7 (Homoginized and heterogeneous potentials)**.** *Under Assumption 1, the following holds.*

i) $\bar{f}$ *is an upper bound on* f*:*

$$\bar{f}(\mathbb{E}_\Theta \mathbf{x}) \geq f(\mathbf{x}) \qquad \text{for any } \mathbf{x} \in \mathcal{F}_\mathcal{X}.$$

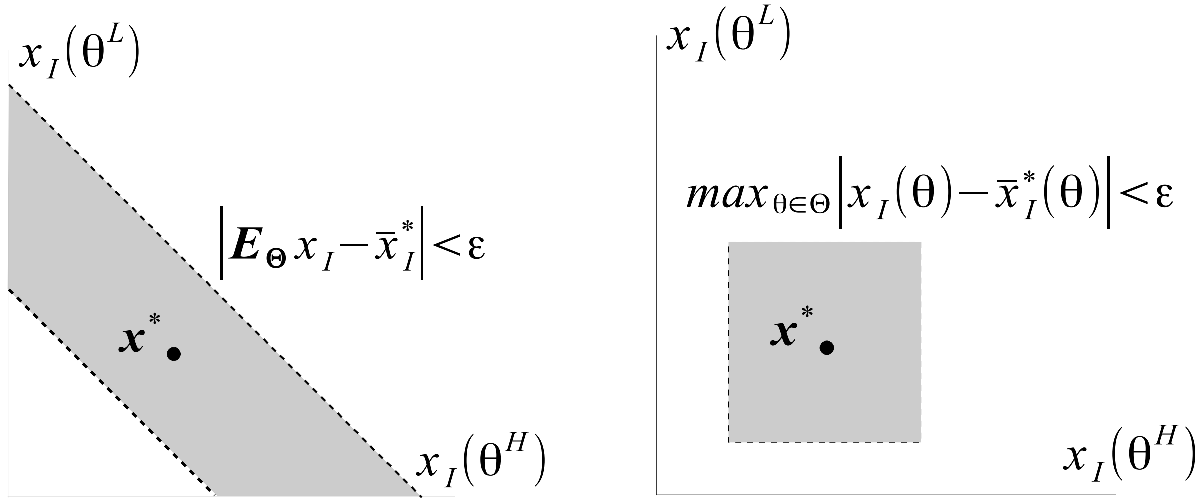
ii) The equality $\bar{f}(\mathbb{E}_\Theta \mathbf{x}) = f(\mathbf{x})$ *holds if and only if* $\mathbf{x}$ *is a Bayesian equilibrium.*

iii) Let $\bar{\mathbf{x}}^* = \mathbb{E}_\Theta \mathbf{x}^*$*. Then,* $\mathbf{x}^*$ *attains a local strict maximum of* f *if and only if* $\bar{\mathbf{x}}^*$ *attains a local strict maximum of* $\bar{f}$*.*

Since a local maximum of $\bar{f}$ coincides with the aggregate of a local maximum of f, one might expect that $\bar{f}$ increases when f increases toward the local maximum, as does in an admissible dynamic. However, it does not. We need to be careful about difference in topology on the space of strategy compositions and on the space of aggregate strategies, as illustrated in Figure 4.

Local stability of a Bayesian equilibrium and local maxima of the heterogeneous potential f is defined on the space of Bayesian strategies, or more precisely the composition space $\mathcal{X}$; see Theorem 10 in Appendix C. So, unless it is global, absorption to a local stable equilibrium needs the *Bayesian* strategy to be close to the equilibrium. The second term in the heterogeneous potential function (9) captures the negative entropy of the composition. With the aggregate strategy fixed, the second term in f is maximized if the composition is completely sorted in the sense that, for each action, given the idiosyncratic payoffs of other actions, there is a threshold value of idiosyncratic payoff of the action and all the agents who have a greater idiosyncratic payoff than

[27] If $\mathbb{P}_\Theta$ is a double exponent distribution with noise level μ and thus the homogenized smooth BRD reduces to a logit dynamic, then the latter term becomes the entropy function $-\mu \sum_{a \in \mathcal{A}} \bar{x}_a \ln \bar{x}_a$.



(a) In the space of aggregate strategy, an ε-neighborhood of the aggregate strategy $\bar{\mathbf{x}}^* = \mathbb{E}_\Theta \mathbf{x}^*$ is the set of aggregate strategies $\bar{\mathbf{x}} \in \Delta^2$ such that $|\bar{\mathbf{x}} - \bar{\mathbf{x}}^*|_2^\infty = \max\{|\bar{x}_I - \bar{x}_I^*|, |\bar{x}_O - \bar{x}_O^*|\} < \varepsilon$, which reduces to $|\bar{x}_I - \bar{x}_I^*| < \varepsilon$. In terms of Bayesian strategy $\mathbf{x} = (\mathbf{x}(\theta^H), \mathbf{x}(\theta^L))$, this is equivalent to $|(x_I(\theta^H) + x_I(\theta^L)) - (x_I^*(\theta^H) + x_I^*(\theta^L))| < \varepsilon$.

(b) In the space of strategy composition, an ε-neighborhood of the Bayesian strategy $\mathbf{x}^*$ is the set of Bayesian strategies $\mathbf{x} = (\mathbf{x}(\theta^H), \mathbf{x}(\theta^L)) \in \Delta^2 \times \Delta^2$ such that $\|\mathbf{x} - \mathbf{x}^*\|_{\mathcal{A}\times\Theta}^\infty = \max\{|\mathbf{x}(\theta^H) - \mathbf{x}^*(\theta^H)|, |\mathbf{x}(\theta^L) - \mathbf{x}^*(\theta^L)|\} < \varepsilon$, which reduces to $\max\{|x_I(\theta^H) - x_I^*(\theta^H)|, |x_I(\theta^L) - x_I^*(\theta^L)|\} < \varepsilon/2$.

Figure 4: Comparison of topology on the space of aggregate strategy and on the space of strategy composition. Here we consider a binary aggregate game with two types $\Theta = \{\theta^H, \theta^L\}$ and consider a neighborhood of $\mathbf{x}^*$; assume that the mass of agents of each type is just a half of the whole population.

the threshold take this action while the others do not.[28] Actually, the strategy composition in a Bayesian equilibrium (i.e., an equilibrium composition) is a completely sorted composition with the aggregate strategy being in aggregate equilibrium.

The homogenized potential $\bar{f}$ does not indeed serve as a Lyapunov function in nonaggregable dynamics. Let $\bar{\mathbf{x}}^*$ be an isolated aggregate equilibrium. Then, $\bar{f}$ attains a local maximum at $\bar{\mathbf{x}}^*$; If the dynamic is aggregable, the aggregate strategy should stay there and $\bar{f}$ should remain at the local maximum, whenever aggregate strategy $\bar{\mathbf{x}}$ reaches $\bar{\mathbf{x}}^*$. But, if the underlying Bayesian strategy $\mathbf{x}_0$ at time 0 is not a Bayesian equilibrium, Lemma 7 implies that $f(\mathbf{x}_0) < \bar{f}(\bar{\mathbf{x}}^*) = f(\mathbf{x}^*)$ where $\mathbf{x}^*$ is the corresponding Bayesian equilibrium. Hence, the heterogeneous potential f is not maximized at $\mathbf{x}_0$ even locally; actually a change in the Bayesian strategy toward $\mathbf{x}^*$ while keeping $\bar{\mathbf{x}} = \mathbb{E}_\Theta \mathbf{x}$ increases the value of f. As f is indeed a strictly increasing Lyapunov function on $\mathcal{F}_\mathcal{X}$ under a Bayesian dynamic, the fact that f is not maximized at $\mathbf{x}_0$ implies that f still increases over time. So $\mathbf{x}$ leaves $\mathbf{x}_0$. On the other hand, since $\bar{f}$ is locally maximized at $\bar{\mathbf{x}}^* = \mathbb{E}_\Theta \mathbf{x}_0$, any move from $\mathbf{x}_0$ decreases $\bar{f}$ at least temporarily. Therefore, the homogenized potential $\bar{f}$ does not tell whether $\bar{\mathbf{x}}$ is settled to equilibrium or not; thus $\bar{f}$ is not a Lyapunov function for the dynamic of aggregate strategy in the heterogeneous setting.

[28] It is indeed the Bayesian strategy that achieves the minimum in the second term of (10) to define the homogenized potential $\bar{f}$. See the proof of Lemma 7.

6 Observational dynamics

In some of major evolutionary dynamics, an agent observes other agents' actions and the observation influences the switching decision—for example, an agent may imitate other agents' actions or the switching rate may depend on the relative payoffs compared to the average payoff of the observed population. We can generalize these dynamics as *observational dynamics* by having the action distribution among observed agents $\tilde{\mathbf{x}} \in \Delta^A$, not only on payoff vector $\boldsymbol{\pi} \in \mathbb{R}^A$, in the argument of the switching rate function $\mathbf{R}$.

Example 9. In a class of **excess payoff dynamics**, a revising agent calculates the average payoff $\tilde{\mathbf{x}} \cdot \boldsymbol{\pi}$ and switches to action j with the rate that increases with the excess payoff of the new action $\pi_j - \tilde{\mathbf{x}} \cdot \boldsymbol{\pi}$. In particular, the switching rate function $R_{ij}(\boldsymbol{\pi}, \tilde{\mathbf{x}}) = [\pi_j - \tilde{\mathbf{x}} \cdot \boldsymbol{\pi}]_+$ defines the **Brown-von Neumann-Nash (BNN) dynamic** (Hofbauer, 2001). ∎

Example 10. In a class of **imitative dynamics**, a revising agent randomly picks another agent and switches to the observed agent's action j with the rate $I_{ij}(\boldsymbol{\pi}) \in \mathbb{R}_+$: the overall switching rate is $R_{ij}(\boldsymbol{\pi}, \tilde{\mathbf{x}}) = \tilde{x}_j I_{ij}(\boldsymbol{\pi})$. There are several imitative protocols that yield the **replicator dynamic** (Taylor and Jonker, 1978): imitative pairwise comparison $I_{ij} = [\pi_j - \pi_i]_+$ (Schlag, 1998), imitation driven by dissatisfaction $I_{ij} = D - \pi_i$ with constant $D \in \mathbb{R}$ (Björnerstedt and Weibull, 1996), and imitation of success $I_{ij} = \pi_j - S$ with constant $S \in \mathbb{R}$ (Hofbauer, 1995a).[29] ∎

They fall into continuous dynamics and satisfy Assumption 2.[30] (Note that Assumption 3 is not needed for continuous dynamics.) Nonaggregability is still confirmed by Theorem 2 for both excess payoff dynamics and imitative dynamics, since the switching rate function R_{ij} is a strictly increasing function of the payoff gain from switch.[31]

We can readily extend all the positive results, i.e., Theorems 1, 4 and 5 and corollary 2, to observational dynamics, if we assume that an agent observes the action distribution of the same type: a type-$\boldsymbol{\theta}$ agent observes $\mathbf{x}(\boldsymbol{\theta}) \in \Delta^A$.[32] This assumption of within-type observability matches with an assumption on imitative dynamics of the society of finitely many subpopulations where a member of each subpopulation imitates the behavior of those in the same subpopulation. The proofs of these theorems in Appendices are indeed written explicitly to include $\mathbf{x}(\boldsymbol{\theta})$ as an argument of switching rate function $\mathbf{R}$.

[29] Sawa and Zusai (2014) verify that any long-run outcome in imitative dynamics is robust to heterogeneity in "aspiration levels," such as D or S. This extends to payoff heterogeneity.

[30] Precisely for observational dynamics, $\bar{R}$ is an upper bound on $R_{ij}(\mathbf{F}[\bar{\mathbf{m}}](\boldsymbol{\theta}), m(\boldsymbol{\theta}))$.

[31] As the easiest case, consider a binary aggregate game and let every agent observe $\tilde{\mathbf{x}} = \bar{\mathbf{x}}$; thus every agent faces the same action distribution of samples. The unconditional total switching rate $R_{IO} + R_{OI}$ increases with $\pi_* - \boldsymbol{\pi} \cdot \bar{\mathbf{x}}$ in excess payoff dynamics. For imitative dynamics, the payoff monotonicity of R_{ij} is implied by that of I_{ij} and indeed satisfied by any of the three protocols that induce the replicator dynamic. While allowing only finitely many types of agents, Zusai (2016) proves that stability of Nash equilibrium in a contractive game is retained in the presence of heterogeneity in payoffs and revision protocols as long as agents follow excess payoff dynamics or admissible non-observational dynamics.

[32] Corollary 1 holds for excess payoff dynamics. Imitative dynamics such as the replicator dynamic satisfy the best response stationarity only if $\mathbf{x}^0$ is in the interior of Δ^A, and thus these corollaries hold for imitative dynamics only in the interior of $\mathcal{F}_{\mathcal{X}}$.

To maintain existence of a unique solution trajectory (Theorem 1) and stationarity of Bayesian equilibrium (Theorem 4), this assumption of within-type observability can be replaced with an alternative assumption that an agent observes the aggregate strategy $\bar{\mathbf{x}}$ instead of $\mathbf{x}(\boldsymbol{\theta})$. But, then the positive correlation (Theorem 5) may not be extended from the homogeneous setting to the heterogeneous setting. If the type of other agents are not distinguishable for an agent and observations are sampled from the entire population, stability analysis becomes essentially different from how we have investigated stability in this paper.[33]

7 Concluding remarks

In this paper, we extend evolutionary dynamics to allow (possibly) continuously many payoff types and persistent payoff heterogeneity. With rigorous formulation as a dynamic in the space of probability measure, we verify the existence of a unique solution path from an arbitrary initial state. Nonaggregability is presented for a general class of evolutionary dynamics, including pairwise comparison dynamics, tempered BRDs, excess payoff dynamics and imitative dynamics. When payoff heterogeneity is persistent, such a dynamic may leave an aggregate equilibrium even if it is stable when the heterogeneity was only transitional. Yet, we can retain stationarity of equilibrium by switching the attention to strategy composition, i.e., the joint distribution of types and actions. Moreover, in a potential game, the *set* of locally stable equilibria in any of such heterogeneous dynamic coincides with that in the homogenized smooth BRD, whose aggregate dynamic is independent of the strategy composition over different types.

In applications to dynamic implementation of the social optimum, dependency of aggregate dynamic on the underlying composition suggests that a bang-bang control results in excessive instability generally in the heterogeneous setting, though it achieves the fastest convergence in the homogeneous setting. Yet, the dynamic Pigouvian pricing, proposed by Sandholm (2005, 2002), still guarantees convergence to the social optimum, while not requiring any ex-ante information about the underlying dynamic or type distribution. Nevertheless, there might be some better scheme that lies between bang-bang controls and the dynamic Pigovian pricing and achieves faster convergence than the Pigovian pricing while not requiring too much information. Actually, nonaggregability also suggests that the direction of transition in aggregate state is related with the underlying strategy composition. If we can find a way to extract the information of the strategy composition from the transition of the aggregate state, it could be used to improve the pricing scheme.[34]

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[33] About unobservable heterogeneity in aspiration levels in imitative dynamics, Sawa and Zusai (2014) verify that, although the dynamic becomes more complicated and basic properties such as positive correlation do not hold, long-run limit outcomes are robust to the introduction of unobservable heterogeneity. For nonobservational dynamics, Zusai (2016) widely extends Nash stability in contractive (stable/negative definite) games and local stability of a (regular) evolutionary stable state to allow unobservable heterogeneity both in payoff functions and switching rate functions, assuming finitely many types and paying attentions to "gains" from revising an action.

[34] In the situation where payoff heterogeneity is not additively separable and the social planner does not exactly know its distribution, Fujishima (2012) proposes a modified Pigouvian pricing that is combined with estimation of the

However, in the heterogeneous dynamics, it depends on the initial composition—not only on the aggregate strategy—*which* aggregate equilibrium is eventually reached in the long run from a given initial state. Now we know that, under a non-aggregable dynamic, the aggregate strategy may escape even from an aggregate equilibrium that is stable under an aggregable dynamic. However, in a potential function, this does not change the set of locally stable equilibria. Therefore, we can use nonaggrgability to select equilibria by requiring robustness of stability to any unsorted distortion in strategy composition. Zusai (2017a) explores this idea by presenting more detailed analysis of nonaggregability and developing the idea of robust critical mass that we briefly mentioned in Section 4, while focusing on a binary ASAG.

Acknowledgment

I greatly acknowledge Bill Sandholm, Marek Weretka, and Noah Williams for encouragement and advice, as well as Man Wah Cheung, Dimitrios Diamantaras, Ian Dobson, Shota Fujishima, Hon Ho Kwok, Catherine Maclean, Daisuke Oyama, Ryoji Sawa and Douglas Webber for their comments. I also thank Olena Berchuk for careful reading and editing suggestions. Finally, I am very grateful to valuable comments from the editor in charge Marciano Siniscalchi and two anonymous referees.

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A Appendix to Sections 2–3

A.1 Measure-theoretic definition of Bayesian strategy

This subsection provides mathematically rigorous definition of Bayesian strategy based on measure theory. Proofs in appendices deal with strategy compositions to properly utilize the Lyapunov stability theorem (Theorem 10 in Appendix C) and borrow the measure-theoretic construction of continuous space evolutionary dynamics such as in Oechssler and Riedel (2001, 2002) and Cheung (2014).

Combination of action profile $\mathfrak{a} : \Omega \to \mathcal{A}$ and type profile $\boldsymbol{\theta} : \Omega \to \Theta$ generates a finite measure $X_a : \mathcal{B}_\Theta \to \mathbb{R}_+$ for each $a \in \mathcal{A}$ from $\mathbb{P}_\Omega$:

$$X_a(B_\Theta) := \mathbb{P}_\Omega(\{\omega \in \Omega \;:\; \mathfrak{a}(\omega) = a \text{ and } \boldsymbol{\theta}(\omega) \in B_\Theta\}) \qquad \text{for each } B_\Theta \in \mathcal{B}_\Theta.$$

$X_a(B_\Theta)$ represents the mass of action-a players whose types are in set B_Θ. X_a is dominated by $\mathbb{P}_\Theta$, denoted by $X_a \ll \mathbb{P}_\Theta$, in the sense that

$$\mathbb{P}_\Theta(B_\Theta) = 0 \quad \Longrightarrow \quad X_a(B_\Theta) = 0 \qquad \text{for each } B_\Theta \in \mathcal{B}_\Theta. \tag{A.1}$$

It follows by Radon-Nikodym theorem that there exists a $\mathcal{B}_\Theta$-measurable nonnegative function $x_a : \Theta \to \mathbb{R}_+$ such that

$$X_a(B_\Theta) = \int_{B_\Theta} x_a(\boldsymbol{\theta})\mathbb{P}_\Theta(d\boldsymbol{\theta}) \qquad \text{for any } B_\Theta \in \mathcal{B}_\Theta.$$

x_a is the density function of measure X_a. The density is determined uniquely in the sense that, if another measurable function x'_a satisfies $X_a(B_\Theta) = \int_{B_\Theta} x'_a(\boldsymbol{\theta})\mathbb{P}_\Theta(d\boldsymbol{\theta})$ for all $B_\Theta \in \mathcal{B}_\Theta$, then $x'_a(\boldsymbol{\theta}) = x_a(\boldsymbol{\theta})$ for $\mathbb{P}_\Theta$-almost all $\boldsymbol{\theta} \in \Theta$.

The distribution of strategies over different types is represented by $\mathbf{X} = (X_a)_{a \in \mathcal{A}} : \mathcal{B}_\Theta \to \Delta^A$, which we call **strategy composition**. We can see this vector measure as a joint probability measure over the product space $\mathcal{A} \times \Theta$. [35] Let $\mathcal{X}$ be the set of strategy compositions, i.e., the set of probability measures over $\mathcal{A} \times \Theta$ that is dominated by $\mathbb{P}_\Theta$ in the above sense.

The Radon-Nikodym density $\mathbf{x} = (x_a)_{a \in \mathcal{A}} : \Theta \to \mathbb{R}^A_+$ is the *Bayesian strategy* corresponding to $\mathbf{X}$. We represent the relationship between $\mathbf{X} = (X_a)_{a \in \mathcal{A}}$ and $\mathbf{x} = (x_a)_{a \in \mathcal{A}}$ as in the above integral equation by $\mathbf{X} = \int \mathbf{x} d\mathbb{P}_\Theta$. From the fact that $\sum_{a \in \mathcal{A}} X_a(B_\Theta) = \mathbb{P}_\Theta(B_\Theta)$ and $X_a(B_\Theta) \geq 0$ for any B_Θ and $a \in \mathcal{A}$, we can confirm that $\mathbf{x}(\boldsymbol{\theta})$ is a probability vector for almost all types:

$$\mathbf{x}(\boldsymbol{\theta}) \in \Delta^A \qquad \text{for } \mathbb{P}_\Theta\text{-almost all } \boldsymbol{\theta} \in \Theta.$$

A Bayesian strategy is ($\mathbb{P}_\Theta$-almost) uniquely determined from a strategy composition by Radon-Nikodym theorem, and vice versa. So, $\mathcal{X}$ is equivalent to the set of Bayesian strategies $\mathcal{F}_\mathcal{X}$.

[35] With abuse of notation, we could say that $\mathbf{X}$ defines the measure of a Borel set $B_{\mathcal{A}\times\Theta}$ on the product space $\mathcal{A} \times \Theta$ by

$$\mathbf{X}(B_{\mathcal{A}\times\Theta}) := \sum_{a \in \mathcal{A}} X_a(\{\boldsymbol{\theta} \in \Theta : (a, \boldsymbol{\theta}) \in B_{\mathcal{A}\times\Theta}\}) = \mathbb{P}_\Omega(\{\omega \in \Omega : (\mathfrak{a}(\omega), \boldsymbol{\theta}(\omega)) \in B_{\mathcal{A}\times\Theta}\}).$$

We call strategy composition $\mathbf{X} \in \mathcal{X}$ an **equilibrium composition**, if

$$\mathbb{P}_\Theta(\beta_a^{-1}(\bar{\mathbf{x}}) \cap B_\Theta) \leq X_a(B_\Theta) \leq \mathbb{P}_\Theta(b_a^{-1}(\bar{\mathbf{x}}) \cap B_\Theta) \quad \text{with } \bar{\mathbf{x}} = \mathbf{X}(\Theta) \tag{A.2}$$

for all $a \in \mathcal{A}$ and $B_\Theta \in \mathcal{B}_\Theta$. This condition is obtained by aggregation of Bayesian equilibrium condition (2) on $x_a(\boldsymbol{\theta})$ over $\boldsymbol{\theta} \in B_\Theta$. Among types in B_Θ, all those who have a as the unique best response must choose this action a in equilibrium and thus $X_a(B_\Theta)$ must be at least $\mathbb{P}_\Theta(\beta_a^{-1}(\bar{\mathbf{x}}) \cap B_\Theta)$. On the other hand, those who have a as one of the best responses may or may not add to action-a players and thus $X_a(B_\Theta)$ is at most $\mathbb{P}_\Theta(b_a^{-1}(\bar{\mathbf{x}}) \cap B_\Theta)$. $\mathbf{X}$ being an equilibrium composition (A.2) is equivalent to its density $\mathbf{x}$ being a Bayesian equilibrium (2).

A.2 Topology of the space of strategy compositions

Choice of topology is a sensitive issue when we argue dynamic of probability measure on a continuous space. We follow the convention in the literature on evolutionary dynamics on a continuous strategy space, such as in Cheung (2014). That is, we use *strong topology* for existence of a solution path and *weak topology* for stability of equilibrium composition.

Below we define these two topologies on the space of finite signed measures $\mathcal{M}_{\mathcal{A}\times\Theta}$. Note that $\mathcal{X} \subset \mathcal{M}_{\mathcal{A}\times\Theta}$ and that $\mathcal{M}_{\mathcal{A}\times\Theta}$ is the tangent space of $\mathcal{X}$. This space $\mathcal{M}_{\mathcal{A}\times\Theta}$ is a vector space and a transition vector stays in this extended (tangent) space.

Strong topology is metrized by the variational norm $\|\cdot\|^\infty_{\mathcal{A}\times\Theta}$ defined as

$$\|\mathbf{M}\|^\infty_{\mathcal{A}\times\Theta} = \sup_{\mathbf{g}} \left\{ \left| \sum_{a\in\mathcal{A}} \int_{\boldsymbol{\theta}\in\Theta} g_a(\boldsymbol{\theta}) M_a(d\boldsymbol{\theta}) \right| \ : \ \sup_{(a,\boldsymbol{\theta})\in\mathcal{A}\times\Theta} |g_a(\boldsymbol{\theta})| \leq 1 \right\},$$

where the first sup is taken over the set of measurable functions $\mathbf{g}$ on $(\mathcal{A}\times\Theta, \mathcal{B}_{\mathcal{A}\times\Theta})$. On the other hand, a Bayesian strategy belongs to $\mathcal{F}_\mathcal{X}$, the space of $\mathcal{B}_\Theta$-measurable vector function from Θ to Δ^A. Note that, if $\mathbf{M} \ll \mathbb{P}_\Theta$, then there uniquely exists a Radon-Nikodym density $\mathbf{m} \in \mathcal{F}_\mathcal{X}$ such that $\mathbf{M} = \int \mathbf{m} d\mathbb{P}_\Theta$ in the sense we defined in Appendix A.1. The theorem below suggests that the variational norm on $\mathcal{X}$ is equivalent to the L^1-norm on $\mathcal{F}_\mathcal{X}$.[36] The proof is provided in Section S2.1 of Supplementary Note.[37]

Theorem 8. *For any finite signed measure $\mathbf{M} \in \mathcal{M}_{\mathcal{A}\times\Theta}$ with density $\mathbf{m} = (m_s)_{s\in\mathcal{A}}$, we have*

$$\|\mathbf{M}\|^\infty_{\mathcal{A}\times\Theta} = \sum_{a\in\mathcal{A}} \mathbb{E}_\Theta |m_a| = \sum_{a\in\mathcal{A}} \int_\Theta |m_a(\boldsymbol{\theta})| \mathbb{P}_\Theta(d\boldsymbol{\theta}). \tag{A.3}$$

With the variational norm, the normed vector space $(\mathcal{M}_{\mathcal{A}\times\Theta}, \|\cdot\|^\infty_{\mathcal{A}\times\Theta})$ is a Banach space; but not with weak topology. By Zeidler (1986, Cor. 3.9), boundedness and Lipschitz continuity of the dynamic in strong topology guarantees the existence and uniqueness of a solution path of the dynamic. See Theorem 9 in Appendix A.3.

[36] Ely and Sandholm (2005) define the standard BRD under payoff heterogeneity directly as a dynamic of $\mathbf{x}$ and adopt L^1 norm.

[37] This density-based formula of the variational norm comes essentially from Theorem 5 in Oechssler and Riedel (2001).

Under **weak topology** on the set of measures over space $\mathcal{S}$, a mapping from $\mathcal{M}(\mathcal{S}) \to \mathbb{R}$ such as $\mu \mapsto \int_{\mathcal{S}} f d\mu$ is continuous for any bounded and continuous function $f : \mathcal{S} \to \mathbb{R}$. In our model, the space $\mathcal{S} := \mathcal{A} \times \Theta$ is separable with metric $d_{\mathcal{A}\times\Theta} : (\mathcal{A} \times \Theta)^2 \to \mathbb{R}_+$ such that[38]

$$d_{\mathcal{A}\times\Theta}((a, \boldsymbol{\theta}), (a', \boldsymbol{\theta}')) := \mathbf{1}\{a \neq a'\} + |\boldsymbol{\theta} - \boldsymbol{\theta}'|^{\infty}_{A}.$$

Then, weak topology is metrized by Prokhorov metric $d_{\mathcal{M}} : {\mathcal{M}_{\mathcal{A}\times\Theta}}^2 \to \mathbb{R}_+$ such that[39]

$$\begin{aligned} d_{\mathcal{M}}(\mathbf{M}, \mathbf{M}') := \inf\{\varepsilon > 0 \ : \ &\mathbf{M}(B_{\mathcal{A}\times\Theta}) \leq \mathbf{M}'(B^{\varepsilon}_{\mathcal{A}\times\Theta}) + \varepsilon \\ &\text{and } \mathbf{M}'(B_{\mathcal{A}\times\Theta}) \leq \mathbf{M}(B^{\varepsilon}_{\mathcal{A}\times\Theta}) + \varepsilon \quad \text{for all } B_{\mathcal{A}\times\Theta} \in \mathcal{B}_{\mathcal{A}\times\Theta}\}, \end{aligned}$$

where $B^{\varepsilon}_{\mathcal{A}\times\Theta}$ is defined from $B_{\mathcal{A}\times\Theta}$ as $B^{\varepsilon}_{\mathcal{A}\times\Theta} := \{(a, \boldsymbol{\theta}) \in \mathcal{A} \times \Theta \ : \ d_{\mathcal{A}\times\Theta}((a, \boldsymbol{\theta}), (a', \boldsymbol{\theta}')) < \varepsilon$ with some $(a', \boldsymbol{\theta}') \in B_{\mathcal{A}\times\Theta}\}$.[40] Under the weak topology, the space of probability measures, i.e., the space of strategy compositions becomes compact. Then, we can apply the Lyapunov stability theorem, as in Cheung (2014, Thm. 6). See Theorem 10 in Appendix C.

A.3 Sketch of Proof of Theorem 1

We prove existence of a unique solution trajectory of a Bayesian dynamic by verifying it for the corresponding dynamic of strategy composition, appealing to the equivalence between Bayesian strategies and strategy compositions. Here we sketch the outline of the proof, while the complete presentation of the proof is provided in Section S2.2 of Supplementary Note.[41]

First, we define the mean dynamic of strategy composition $\mathbf{V} = (V_i)_{i\in\mathcal{A}} : \mathcal{X} \times \mathcal{C}_{\Theta} \to \mathcal{M}_{\mathcal{A}\times\Theta}$ from the Bayesian mean dynamic (4) as

$$\begin{aligned} \dot{X}_i(B_{\Theta}) = V_i[\mathbf{X}, \boldsymbol{\pi}](B_{\Theta}) &= \int_{B_{\Theta}} v_i[\boldsymbol{\pi}(\boldsymbol{\theta}), \mathbf{x}(\boldsymbol{\theta})]\mathbb{P}_{\Theta}(d\boldsymbol{\theta}) \\ &= \int_{B_{\Theta}} \sum_{j\in\mathcal{A}} R_{ji}(\boldsymbol{\pi}(\boldsymbol{\theta}), \mathbf{x}(\boldsymbol{\theta}))X_j(d\boldsymbol{\theta}) - \int_{B_{\Theta}} \left\{ \sum_{j\in\mathcal{A}} R_{ij}(\boldsymbol{\pi}(\boldsymbol{\theta}), \mathbf{x}(\boldsymbol{\theta})) \right\} X_i(d\boldsymbol{\theta}) \end{aligned} \tag{A.4}$$

for each $B_{\Theta} \in \mathcal{B}_{\Theta}$, given strategy composition $\mathbf{X} \in \mathcal{X}$ and payoff function $\boldsymbol{\pi} : \Theta \to \mathbb{R}^A$. In short, we write $\dot{\mathbf{X}} = \mathbf{V}[\mathbf{X}, \boldsymbol{\pi}]$.

In a population game $\mathbf{F} : \Delta^A \times \Theta \to \mathbb{R}^A$, the mean dynamic (A.4) of strategy composition defines an autonomous dynamic $\mathbf{V}^{\mathbf{F}}$ over $\mathcal{X}$ by

$$\dot{\mathbf{X}} = \mathbf{V}^{\mathbf{F}}[\mathbf{X}] := \mathbf{V}[\mathbf{X}, \mathbf{F}[\mathbf{X}(\Theta)]] \in \mathcal{M}_{\mathcal{A}\times\Theta}$$

[38]The metric $d_{\mathcal{A}\times\Theta}$ is a product metric constructed from the discrete norm on $\mathcal{A}$ and the sup norm on $\Theta \subset \mathbb{R}^A$. Notice $A < \infty$; so the product metric $d_{\mathcal{A}\times\Theta}$ makes $\mathcal{A} \times \Theta$ separable. Here $\mathbf{1}\{a \neq a'\}$ is an indicator function and takes 1 if $a \neq a'$ and 0 otherwise.

[39]If there is no payoff heterogeneity, i.e., $\Theta = \{\boldsymbol{\theta}_0\}$, then composition $\mathbf{M}$ can be simply represented by an A-dimensional vector $(\bar{m}_a)_{a\in\mathcal{A}} \in \mathbb{R}^A$ such that $\bar{m}_a = M_a(\{\boldsymbol{\theta}_0\})$. Then, $d_{\mathcal{M}}(\mathbf{M}, \mathbf{M}') = \varepsilon$ is equivalent to $\sup_{a\in\mathcal{A}} |\bar{m}_a - \bar{m}_{a'}| = \varepsilon$. So the metric $d_{\mathcal{M}}$ reduces to the sup norm on $\mathbb{R}^A$.

[40]If $\varepsilon < 1$, the condition for $(a, \boldsymbol{\theta}) \in B^{\varepsilon}_{\mathcal{A}\times\Theta}$ is equivalent to the existence of $\boldsymbol{\theta}' \in \Theta$ such that $|\boldsymbol{\theta} - \boldsymbol{\theta}'|^{\infty}_{A} < \varepsilon$ and $(a, \boldsymbol{\theta}') \in B_{\mathcal{A}\times\Theta}$.

[41]Basically this proof follows the proof of Lipschitz continuity of a continuous-strategy evolutionary dynamic, as in Cheung (2014): both deal with the dynamic of probability measure on a (possibly) continuous space. However, there are two technical differences: the need for Lebesgue decomposition of an arbitrary finite signed measure and the discontinuity of switching rates in exact optimization protocols. See footnotes 43 and 47.

for each strategy composition $\mathbf{X} \in \mathcal{X}$. Then, $\mathbf{V^F}[\mathbf{X}](B_\Theta) = \int_{B_\Theta} \mathbf{v^F}[\mathbf{x}](\boldsymbol{\theta})\mathbb{P}_\Theta(d\boldsymbol{\theta})$, where $\mathbf{x}$ is the corresponding Bayesian strategy, i.e., the Radon-Nikodym density of $\mathbf{X}$. Theorem 8 suggests that Lipschitz continuity of $\mathbf{v^F}$ in L^1-norm is equivalent to Lipschitz continuity of $\mathbf{V^F}$ in the variational norm

To argue unique existence of a solution trajectory, we exploit the known result on a Lipschitz continuous dynamic in a Banach space as in the theorem below.[42]

Theorem 9 (Zeidler, 1986: Corollary 3.9)**.** *Consider a dynamic $\dot{z} = V(z)$ with $V : \mathcal{Z} \to \mathcal{Z}$. If the space $\mathcal{Z}$ is a Banach space and the dynamic V is Lipschitz continuous and bounded, then there exists a unique solution $\{z_t\}_{t\in\mathbb{R}_+}$ from any initial state in $z_0 \in \mathcal{Z}$.*

For this, we need a Banach space. But, the space of strategy composition $\mathcal{X}$ is not a vector space. Thus, we extend the dynamic to the space of finite signed measures $\mathcal{M}_{\mathcal{A}\times\Theta}$. Since the mean dynamic $\mathbf{V}[\mathbf{X}, \boldsymbol{\pi}](B_\Theta)$ is defined as the aggregate transition of Bayesian strategy, i.e., the density of $\mathbf{X}$ over types in B_Θ, we still need a density of a measure on this extended space. However, a finite signed measure may not be absolute continuous with respect to the type distribution $\mathbb{P}_\Theta$. We use the Lebesgue decomposition theorem (Lemma 1 in Supplementary Note) to extract the absolute continuous part:[43] a finite signed measure $\mathbf{M}$ is decomposed to a $\mathbb{P}_\Theta$-absolute continuous measure $\tilde{\mathbf{M}} \ll \mathbb{P}_\Theta$ and the orthogonal part $\hat{\mathbf{M}} \perp \mathbb{P}_\Theta$.[44] The absolutely continuous part $\tilde{\mathbf{M}}$ has density $\tilde{\mathbf{m}}$ with respect to $\mathbb{P}_\Theta$. Let $\tilde{\mathcal{M}}_{\mathcal{A}\times\Theta}$ be the space of $\mathbb{P}_\Theta$-absolute continuous measures. Besides, the orthogonality of $\hat{\mathbf{M}}$ implies $\|\tilde{\mathbf{M}}\| \leq \|\mathbf{M}\|$.

Yet, a density function $\tilde{\mathbf{m}}$ of $\tilde{\mathbf{M}}$ may not be bounded, while that of a strategy composition, i.e., a Bayesian strategy $\mathbf{x}$ is bounded in the sense that $\mathbf{x}(\boldsymbol{\theta})$ of almost every type $\boldsymbol{\theta}$ belongs to a bounded set Δ^A. As we will utilize the assumptions that the payoff function and the switching rate function (the conditional switch rate function $Q_{\cdot\cdot}$ for exact optimization dynamics) are continuous and thus bounded if its domain is restricted to a compact set, we truncate $\tilde{\mathbf{m}}$ by a rounding function $\boldsymbol{\mu} = (\mu_j)_{j\in\mathcal{A}} : \mathbb{R}^A \to [-3,3]^A$ such that $\boldsymbol{\mu}(\mathbf{z}) = \mathbf{z}$ if $\mathbf{z} \in \Delta^A$ and $\boldsymbol{\mu}$ is Lipschitz continuous with constant $L_{\boldsymbol{\mu}}$.[45]

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[42] Ely and Sandholm (2005, Theorem A.3.) also guarantee the existence of a unique solution trajectory for a Bayesian dynamic on $\mathcal{F}_\mathcal{X}$ with L^1-norm from Lipschitz continuity of the dynamic.

[43] This need for Lebesgue decomposition is the first difference from evolutionary dynamics on a continuous strategy space, which can be directly defined for the transition of a probability measure—a strategy distribution over the continuous strategy space—without having a density. It comes from the difference in an individual agent's characteristic and decision between these two kinds of (possibly) continuous evolutionary dynamics. In our heterogeneous evolutionary dynamics, an agent is characterized by its type—the type of an agent never changes and the type affects the switching rate through payoff heterogeneity: the first nature implies $\mathbb{P}_\Theta \gg \mathbf{X}$ and the second nature makes it indeed natural to construct the dynamic from describing the switches of each type of agents and thus defining the transition of a Bayesian strategy, namely the density function. On the other hand, in continuous-strategy evolution dynamics, an agent is assumed to be homogeneous and thus has no inherited characteristic; thus there is no *ad hoc* distribution that should dominate the strategy distribution.

[44] Orthogonality of $\hat{\mathbf{M}} = (\hat{M}_a)_{a\in\mathcal{A}} \in \mathcal{M}_{\mathcal{A}\times\Theta}$, i.e., $\hat{\mathbf{M}} \perp \mathbb{P}_\Theta$ means that, for each $a \in \mathcal{A}$, there exists $E_a \in \mathcal{B}_\Theta$ such that $\hat{M}_a(B_\Theta \cap E_a) = 0$ and $\mathbb{P}_\Theta(B_\Theta \setminus E_a) = 0$ for any $B_\Theta \in \mathcal{B}_\Theta$.

[45] For example, define $\mu^0 : \mathbb{R} \to [-3,3]$ such as $\mu^0(z) := -3 + \exp(z+2)$ for $z < -2$, $\mu^0(z) := z$ for $z \in [-2,2]$ and $\mu^0(z) := 3 - \exp(2-z)$ for $z > 2$. Then, define vector function $\boldsymbol{\mu} = (\mu_a)_{a\in\mathcal{A}}$ by $\mu_a(\mathbf{z}) = \mu^0(z_a)$ for each $a \in \mathcal{A}$ and $\mathbf{z} \in \Delta^A$.

Then, we redefine the function $\mathbf{v}^{\mathbf{F}} : \tilde{\mathcal{M}}_{\mathcal{A}\times\Theta} \times \Theta \to \mathbb{R}$ on the extended domain by

$$v_i^{\mathbf{F}}[\tilde{\mathbf{M}}](\boldsymbol{\theta}) := \sum_{j\in\mathcal{A}} R_{ji}(\mathbf{F}[\boldsymbol{\mu}(\tilde{\mathbf{M}}(\Theta))](\boldsymbol{\theta}), \boldsymbol{\mu}(\tilde{\mathbf{m}}(\boldsymbol{\theta})))\mu_j(\tilde{\mathbf{m}}(\boldsymbol{\theta})) \\ - \sum_{j\in\mathcal{A}} R_{ij}(\mathbf{F}[\boldsymbol{\mu}(\tilde{\mathbf{M}}(\Theta))](\boldsymbol{\theta}), \boldsymbol{\mu}(\tilde{\mathbf{m}}(\boldsymbol{\theta})))\mu_i(\tilde{\mathbf{m}}(\boldsymbol{\theta}))$$

for each $i \in \mathcal{A}$ and any $\mathbb{P}_\Theta$-absolute continuous finite signed vector measure $\tilde{\mathbf{M}} \in \tilde{\mathcal{M}}_{\mathcal{A}\times\Theta}$. Here $\tilde{\mathbf{m}}$ is the Radon-Nikodym density of $\tilde{\mathbf{M}}$. This leads to the extension of $\mathbf{V}^{\mathbf{F}} := (V_i^{\mathbf{F}})_{i\in\mathcal{A}}$ to $\mathcal{M}_{\mathcal{A}\times\Theta}$, such as

$$V_i^{\mathbf{F}}[\mathbf{M}](B_\Theta) = \int_{B_\Theta} v_i^{\mathbf{F}}[\tilde{\mathbf{M}}](\boldsymbol{\theta})\mathbb{P}_\Theta(d\boldsymbol{\theta}) \qquad \text{for any } B_\Theta \in \mathcal{B}_\Theta$$

for each $i \in \mathcal{A}$ and any finite signed vector measure $\mathbf{M} \in \mathcal{M}_{\mathcal{A}\times\Theta}$, where $\tilde{\mathbf{M}}$ is the $\mathbb{P}_\Theta$-absolute continuous part of $\mathbf{M}$ in the Lebesgue decomposition of $\mathbf{M}$. As only this part matters to the value of $\mathbf{V}^{\mathbf{F}}$, we have $\mathbf{V}^{\mathbf{F}}[\mathbf{M}] = \mathbf{V}^{\mathbf{F}}[\tilde{\mathbf{M}}]$.

To prove Lipschitz continuity of $\mathbf{V}^{\mathbf{F}}$, we first look at $\mathbf{V}^{\mathbf{F}}$ on $\tilde{\mathcal{M}}_{\mathcal{A}\times\Theta}$: in Supplementary Note S2.2, we find $L_V^{\mathbf{F}} > 0$ such that[46]

$$\|\mathbf{V}^{\mathbf{F}}[\tilde{\mathbf{M}}] - \mathbf{V}^{\mathbf{F}}[\tilde{\mathbf{M}}']\| \leq L_V^{\mathbf{F}}\|\tilde{\mathbf{M}} - \tilde{\mathbf{M}}'\| \qquad \text{for any } \tilde{\mathbf{M}}, \tilde{\mathbf{M}}' \in \tilde{\mathcal{M}}_{\mathcal{A}\times\Theta}. \tag{A.5}$$

Then, this implies Lipschitz continuity over the whole space $\mathcal{M}_{\mathcal{A}\times\Theta}$, because $\|\tilde{\mathbf{M}} - \tilde{\mathbf{M}}'\| \leq \|\mathbf{M} - \mathbf{M}'\|$ for any $\mathbf{M} = \tilde{\mathbf{M}} + \hat{\mathbf{M}}, \mathbf{M}' = \tilde{\mathbf{M}}' + \hat{\mathbf{M}}' \in \mathcal{M}_{\mathcal{A}\times\Theta}$.

For continuous dynamics, the Lipschitz continuity of $\mathbf{V}^{\mathbf{F}}$ is a natural consequence of the Lipschitz continuity of the switching rate function $\mathbf{R}$ and of the common payoff function $\mathbf{F}^0$.

On the other hand, the exact optimization protocol is not discontinuous.[47] In particular to the Lipschitz continuity of $\mathbf{V}^{\mathbf{F}}$, the types who switch the best response actions when the strategy composition changes from $\tilde{\mathbf{M}}$ to $\tilde{\mathbf{M}}'$ should experience discontinuous change in the switching rates. However, this discontinuous change in their switching rates is bounded thanks to the boundedness of the switching rate function $\mathbf{R}$. Further, thanks to Assumption 3, the mass of agents who belong to such types expands only (Lipschitz) continuously with the change in the composition.[48] As a result, the aggregate change in their revision rates grows only continuously.

[46] Here the norm $\|\cdot\|$ is the variational norm, defined in Appendix A.2.

[47] This discontinuity in exact optimization dynamics is the second difference from the preceding studies on evolutionary dynamics on continuous strategy space, since these consider the dynamics with continuous switching rates, e.g., the replicator dynamic (Oechssler and Riedel, 2001), the BNN dynamic (Hofbauer, Oechssler, and Riedel, 2009), the gradient dynamic (Friedman and Ostrov, 2013), payoff comparison dynamics (Cheung, 2014) and the logit dynamic (Lahkar and Riedel, 2015). Here, we suppress discontinuity in switching rates by continuity in the mass of agents who experience discontinuous change in switching rates. This is done by adding Assumption 3, i.e., continuity of type distribution. This continuity mitigates discontinuity in switching rates and retains continuity of the dynamic, thanks to the construction of our dynamic that implies $\mathbb{P}_\Theta \gg \mathbf{X}$ as argued in footnote 43.

[48] Note that this assumption also restricts the mass of types who have multiple best responses to be a null set (zero measure) in $\mathbb{P}_\Theta$.

B Appendix to Section 4

B.1 Proof of Theorem 2

Henceforth, we let $R^0_{ij}(\boldsymbol{\theta}) := R^0_{ij}(F[\bar{\mathbf{x}}](\boldsymbol{\theta}))$ and $\hat{R}^0_{ij}(\boldsymbol{\theta}) = R^0_{ij}(\boldsymbol{\theta}) - \mathbb{E}_\Theta R^0_{ij}$. Likewise, let $\hat{X}_i(\tilde{\Theta}) = X_i(\tilde{\Theta}) - \bar{x}_i \mathbb{P}_\Theta(d\boldsymbol{\theta})$.

First of all, recall the Bayesian dynamic:

$$\dot{x}_i(\theta) = \sum_{j\neq i} x_j(\boldsymbol{\theta}) R^0_{ji}(\boldsymbol{\theta}) - x_i(\boldsymbol{\theta}) \sum_{j\neq i} R^0_{ij}(\boldsymbol{\theta}).$$

Aggregating this over all $\boldsymbol{\theta}$, we obtain transition of the aggregate strategy:

$$\begin{aligned}\dot{\bar{x}}_i &= \sum_{j\neq i} \left\{ \int_\Theta R^0_{ji}(\boldsymbol{\theta}) X_j(d\boldsymbol{\theta}) - \int_\Theta R^0_{ji}(\boldsymbol{\theta}) X_i(d\boldsymbol{\theta}) \right\} \\ &= \sum_{j\neq i} \left\{ \bar{x}_j \mathbb{E}_\Theta R_{ji} - \bar{x}_i \mathbb{E}_\Theta R_{ij} + \int_\Theta \hat{R}_{ji}(\boldsymbol{\theta}) \hat{X}_j(d\boldsymbol{\theta}) - \int_\Theta \hat{R}_{ij}(\boldsymbol{\theta}) \hat{X}_i(d\boldsymbol{\theta}) \right\}\end{aligned}$$

Since $\hat{X}_i = -\sum_{k\neq i} \hat{X}_k$, the third term can be rearranged as

$$-\sum_{j\neq i} \int_\Theta \hat{R}_{ij}(\boldsymbol{\theta}) \hat{X}_i(d\boldsymbol{\theta}) = \sum_{j\neq i}\sum_{k\neq i} \int_\Theta \hat{R}_{ij}(\boldsymbol{\theta}) \hat{X}_k(d\boldsymbol{\theta}) = \sum_{j\neq i}\sum_{k\neq i} \int_\Theta \hat{R}_{ik}(\boldsymbol{\theta}) \hat{X}_j(d\boldsymbol{\theta}).$$

Therefore, we have

$$\dot{\bar{x}}_i = \sum_{j\neq i} (\bar{x}_j \mathbb{E}_\Theta R_{ji} - \bar{x}_i \mathbb{E}_\Theta R_{ij}) + \sum_{j\neq i} \int_\Theta \{\hat{R}_{ji}(\boldsymbol{\theta}) + \sum_{k\neq i} \hat{R}_{ik}(\boldsymbol{\theta})\} \hat{X}_j(d\boldsymbol{\theta}).$$

The first term is wholly determined only from the aggregate strategy. It is indeed the transition of the aggregate strategy if the switching rate is constant. The second term is the correlation between the unconditional total switching rate $R_{ji} + \sum_{k\neq i} R_{ik}$ and the strategy composition X_j. If the unconditional total switching rate varies with types, the transition of the aggregate strategy differs with strategy composition through the difference in this correlation term, as we see below.

Proof. Suppose that there exists a pair of two distinctive actions i, j such that $\bar{x}_j > 0$ and the variation of $R^0_{ji}(\boldsymbol{\theta}) + \sum_{k\neq i} R^0_{ik}(\boldsymbol{\theta})$ is not zero. This implies existence of $\Theta^+ \subset \Theta$ such that $\mathbb{P}_\Theta(\Theta^+) > 0$ and

$$\hat{R}^0_{ji}(\boldsymbol{\theta}) + \sum_{k\neq i} \hat{R}^0_{ik}(\boldsymbol{\theta}) > 0 \qquad \text{for all } \boldsymbol{\theta} \in \Theta^+$$

and existence of $\Theta^- \subset \Theta$ such that $\mathbb{P}_\Theta(\Theta^-) > 0$ and

$$\hat{R}^0_{ji}(\boldsymbol{\theta}) + \sum_{k\neq i} \hat{R}^0_{ik}(\boldsymbol{\theta}) < 0 \qquad \text{for all } \boldsymbol{\theta} \in \Theta^-.$$

Note that $\mathbb{P}_\Theta(\Theta^+) \leq 1 - \mathbb{P}_\Theta(\Theta^-) < 1$ and similarly $\mathbb{P}_\Theta(\Theta^+) < 1$.

Given non-pure aggregate strategy $\bar{\mathbf{x}}$, we can find ε such that $\varepsilon = 0.5 \min\{\bar{x}_j, 1 - \bar{x}_i\} > 0$. Define $\mathbf{X}^+$ by

$$X^+_i(\tilde{\Theta}) = (\bar{x}_i + \varepsilon)\mathbb{P}_\Theta(\tilde{\Theta}) - \varepsilon \frac{\mathbb{P}_\Theta(\tilde{\Theta} \cap \Theta^+)}{\mathbb{P}_\Theta(\Theta^+),}$$

$$X_j^+(\tilde{\Theta}) = (\bar{x}_j - \varepsilon)\mathbb{P}_\Theta(\tilde{\Theta}) + \varepsilon\frac{\mathbb{P}_\Theta(\tilde{\Theta}\cap\Theta^+)}{\mathbb{P}_\Theta(\Theta^+),}$$

$$X_k^+(\tilde{\Theta}) = \bar{x}_k\mathbb{P}_\Theta(\tilde{\Theta}) \qquad \text{for all } k \neq i, j.$$

for each $\mathbb{P}_\Theta$-measurable set $\tilde{\Theta} \subset \Theta$.

Then, the correlation term reduces to

$$\int_\Theta \{\hat{R}_{ji}^0(\boldsymbol{\theta}) + \sum_{k\neq i}\hat{R}_{ik}^0(\boldsymbol{\theta})\}\hat{X}_j^+(d\boldsymbol{\theta}) = \varepsilon\left(\frac{1}{\mathbb{P}_\Theta(\Theta^+)} - 1\right)\int_{\Theta^+}\{\hat{R}_{ji}^0(\boldsymbol{\theta}) + \sum_{k\neq i}\hat{R}_{ik}^0(\boldsymbol{\theta})\}\mathbb{P}_\Theta(d\boldsymbol{\theta}) > 0.$$

Therefore, from this strategy composition $\mathbf{X}^+$, the transition of the aggregate strategy is

$$\bar{x}_i > \sum_{j\neq i}(\bar{x}_j\mathbb{E}_\Theta R_{ji} - \bar{x}_i\mathbb{E}_\Theta R_{ij}).$$

We define $\mathbf{X}^-$ by

$$X_i^-(\tilde{\Theta}) = (\bar{x}_i + \varepsilon)\mathbb{P}_\Theta(\tilde{\Theta}) - \varepsilon\frac{\mathbb{P}_\Theta(\tilde{\Theta}\cap\Theta^-)}{\mathbb{P}_\Theta(\Theta^-),}$$

$$X_j^-(\tilde{\Theta}) = (\bar{x}_j - \varepsilon)\mathbb{P}_\Theta(\tilde{\Theta}) + \varepsilon\frac{\mathbb{P}_\Theta(\tilde{\Theta}\cap\Theta^-)}{\mathbb{P}_\Theta(\Theta^-),}$$

$$X_k^-(\tilde{\Theta}) = \bar{x}_k\mathbb{P}_\Theta(\tilde{\Theta}) \qquad \text{for all } k \neq i, j.$$

for each $\mathbb{P}_\Theta$-measurable set $\tilde{\Theta} \subset \Theta$. Then, similarly to the above calculation, we obtain the negative correlation and

$$\bar{x}_i < \sum_{j\neq i}(\bar{x}_j\mathbb{E}_\Theta R_{ji} - \bar{x}_i\mathbb{E}_\Theta R_{ij}).$$

B.2 Proof of Theorem 3

□

Proof. Let $\hat{R}^0 := R_{IO}(\mathbf{F}[\bar{x}_I^*](\hat{\theta}_O^0))$ and $\underline{R}^0 = R_{OI}(\mathbf{F}[\bar{x}_I^*](\underline{\theta}_O))$; then, $r = \underline{R}^0/\hat{R}^0$.

i) Consider an arbitrary type $\theta_O > \hat{\theta}_O^0$. At period 0, while every agent of this type is taking action I in the reversed composition, this type's best response is O and the payoff gain from switch is $\theta_O - F^0(\bar{x}_I^*) > \hat{\theta}_O^0 - F^0(\bar{x}_I^*)$. Thus, the agent switches from I to O at rate $R_{IO}(\mathbf{F}[\bar{x}_I^*](\theta_O)) \geq R_{IO}(\mathbf{F}[\bar{x}_I^*](\hat{\theta}_O^0)) = \hat{R}^0$ by payoff monotonicity of R_{IO}. Therefore, at period 0, the transition of $x_{I,t}(\theta_O)$ follows

$$\dot{x}_{I,0}(\theta_O) = -R_{IO}(\mathbf{F}[\bar{x}_I^*](\theta_O))x_{I,0}(\theta_O) \leq -\hat{R}^0 \qquad \text{for all } \theta_O > \hat{\theta}_O^0.$$

For any type $\theta_O \in (\theta_O^*, \hat{\theta}_O^0]$, the best response is O at period 0 since $\theta_O > \theta_O^* = F^0(\bar{x}_I^*)$. All agents of this type is taking action O at this period. So they do not switch at period 0 and $x_{I,t}(\theta_O)$ of this type remains unchanged at period 0:

$$\dot{x}_{I,0}(\theta_O) = 0 \qquad \text{for all } \theta_O \in (\theta_O^*, \hat{\theta}_O^0].$$

Consider an arbitrary type $\theta_O \leq \theta_O^*$. At period 0, while every agent of this type is taking action O in the reversed composition, this type's best response is I and the payoff gain from switch is $F^0(\bar{x}_I^*) - \theta_O \leq F^0(\bar{x}_I^*) - \underline{\theta}_O$. Thus, the agent switches from O to I at rate $R_{OI}(\mathbf{F}[\bar{x}_I^*](\theta_O)) \leq R_{OI}(\mathbf{F}[\bar{x}_I^*](\underline{\theta}_O)) = \underline{R}^0$ by payoff monotonicity of R_{OI}. Therefore, at period 0, the transition of

$x_{I,t}(\theta_O)$ follows

$$\dot{x}_{I,0}(\theta_O) = R_{OI}(\mathbf{F}[\bar{x}_I^*](\theta_O))x_{O,0}(\theta_O) \le \underline{R}^0 \qquad \text{for all } \theta_O \le \theta_O^*.$$

Aggregating $\dot{x}_{I,0}(\theta_O)$ over all types, we obtain the transition of the aggregate strategy:

$$\begin{aligned}\dot{\bar{x}}_{I,0} &= \int_\Theta \dot{x}_{I,0}(\theta_O)\mathbb{P}_\Theta(d\theta_O)\\ &\le -\hat{R}^0\mathbb{P}_\Theta(\{\theta_O : \theta_O \ge \hat{\theta}_O^0\}) + \underline{R}^0\mathbb{P}_\Theta(\{\theta_O : \theta_O \le \theta_O^*\})\\ &= -\bar{x}_I^*(\hat{R}^0 - \underline{R}^0) = -(1-r)\bar{x}_I^*\hat{R}^0.\end{aligned}$$

The second equality comes from $\mathbb{P}_\Theta(\{\theta_O : \theta_O > \hat{\theta}_O^0\}) = 1 - P(\theta_O^0) = \bar{x}_I^* = P_\Theta(\theta_O^*) = \mathbb{P}_\Theta(\{\theta_O : \theta_O \le \theta_O^*\})$. The last line comes from $r = \underline{R}^0/\hat{R}^0$. Since $\bar{x}_I^*$ is assumed to be positive and the payoff monotonicity implies $\hat{R}^0 = R_{IO}\mathbf{F}[\bar{x}_I^*](\hat{\theta}_O^0)) > 0$ by $\hat{\theta}_O^0 > \theta_O^* = F^0(\bar{x}_I^*)$, we have $\dot{\bar{x}}_{I,0} < 0$ if $r < 1$.

ii) Since F_I^0 increases with $\bar{x}_I$, we have $F_I^0(\bar{x}_{I,t}) \le F_I^0(\bar{x}_I^*) = \theta_O^*$ as long as $\bar{x}_{I,t} \le \bar{x}_I^*$. Under such aggregate strategy $\bar{x}_I \le \bar{x}_I^*$, the best response for an arbitrary type $\theta_O > \theta_O^*$ remains to be action O. Therefore, the transition of $x_{I,t}$ for type $\theta_O > \hat{\theta}_O^0$ still follows

$$\dot{x}_{I,t}(\theta_O) = -R_{IO}(\mathbf{F}[\bar{x}_{I,t}](\theta_O))x_{I,t}(\theta_O) \le -\hat{R}^0 x_{I,t}(\theta_O) \qquad \text{for all } \theta_O > \hat{\theta}_O^0.$$

The inequality is implied by payoff monotonicity, since the payoff gain is $\theta_O - F^0(\bar{x}_{I,t}) \ge \theta_O - F^0(\bar{x}_I^*) > \hat{\theta}_O^0 - F^0(\bar{x}_I^*)$. Therefore, $x_{I,t}(\theta_O)$ follows

$$x_{I,t}(\theta_O) \le \exp(-R_{IO}(\mathbf{F}[\bar{x}_I^*](\hat{\theta}_O^0)t)x_{I,0}(\theta_O) \le \exp(-R_{IO}(\mathbf{F}[\bar{x}_I^*](\hat{\theta}_O^0)t) \qquad \text{for all } \theta_O > \hat{\theta}_O^0.$$

Since all the agents of type $\theta_O \in (\theta_O^*, \hat{\theta}_O^0]$ are taking the best response action O, there is no change in their actions:

$$x_{I,t}(\theta_O) = 0 \qquad \text{for all } \theta_O \in (\theta_O^*, \hat{\theta}_O^0].$$

On the other hand, the best response for type $\theta_O \le \theta_O^*$ may change from I to O. If it changes, $\dot{x}_{I,t}(\theta_O) \le 0$. Even while I is still the best response, the payoff gain decreases as $F^0(\bar{x}_{I,t}) - \theta_O < F^0(\bar{x}_I^*) - \theta_O \le F^0(\bar{x}_I^*) - \underline{\theta}_O$. Thus, the switching rate from O to I is still bounded by $R_{OI}(\mathbf{F}[\bar{x}_{I,t}](\theta_O)) \le R_{OI}(\mathbf{F}[\bar{x}_I^*](\underline{\theta}_O)) = \underline{R}^0$. Therefore, in both cases, the transition of $x_{I,t}(\theta_O)$ satisfies

$$\dot{x}_{I,t}(\theta_O) \le \underline{R}^0 x_{O,t}(\theta_O)$$

and thus we have

$$x_{I,t}(\theta_O) \le \left\{1 - \exp(-\underline{R}^0 t)\right\} x_{I,0}(\theta_O) = 1 - \exp(-\underline{R}^0 t) \qquad \text{for all } \theta_O \le \theta_O^*.$$

Aggregating $x_{I,t}(\theta_O)$ over all types, we obtain the aggregate strategy:

$$\begin{aligned}\bar{x}_{I,t} &= \int_\Theta x_{I,t}(\theta_O)\mathbb{P}_\Theta(d\theta_O)\\ &\le \exp(-\hat{\theta}_O^0 t)\mathbb{P}_\Theta(\{\theta_O : \theta_O \ge \hat{\theta}_O^0\}) + \left\{1 - \exp(-\underline{R}^0 t)\right\}\mathbb{P}_\Theta(\{\theta_O : \theta_O \le \theta_O^*\})\\ &= \bar{x}_I^*\left\{\exp(-\hat{\theta}_O^0 t) + 1 - \exp(-\underline{R}^0 t)\right\} =: \bar{\bar{x}}_{I,t}.\end{aligned}$$

Notice that $\bar{\bar{x}}_{I,0} = \bar{x}_I^*$; further, $r < 1$ implies that $\bar{\bar{x}}_{I,t} < \bar{x}_I^*$ for all $t \in (0, \infty)$ and

$$\frac{d\bar{\bar{x}}_{I,t}}{dt} = \bar{x}_I^* \left\{ -\hat{\theta}_O^0 \exp(-\hat{\theta}_O^0 t) + \underline{R}^0 \exp(-\underline{R}^0 t) \right\} \begin{cases} < 0 & \text{if } t < T \\ = 0 & \text{if } t = T \\ > 0 & \text{if } t > T. \end{cases}$$

At time T, this upper bound reaches $\bar{\bar{x}}_{I,T} = \bar{x}_I^*(1 - (1-r)r^{r/(1-r)}) < \bar{x}_I^*$. Since $\bar{x}_{I,T} \leq \bar{\bar{x}}_{I,T} < \bar{x}_I^* = \bar{x}_{I,0}$ and $\bar{x}_{I,t}$ is continuous in t, it must have hit $\bar{\bar{x}}_{I,T}$ at some $t \in (0, T]$.

The above calculation presumes $\bar{x}_{I,t} \leq \bar{x}_I^*$. From part i), we find this holds when t is close to 0. Suppose that $\bar{x}_{I,t}$ come back to $\bar{x}_I^*$ at some later $t < \infty$ and let $T' > 0$ be the first time such that $\bar{x}_{I,T'} = \bar{x}_I^*$. Then, since $\bar{x}_{I,t} \leq \bar{x}_I^*$ for all $t \in [0, T']$, the above calculation holds thus we have $\bar{x}_{I,t} \leq \bar{\bar{x}}_{I,t}$ for all $t \in [0, T']$. In particular, we have $\bar{x}_{I,T'} = \bar{x}_I^* \leq \bar{\bar{x}}_{I,T'}$. However, this contradicts with $\bar{\bar{x}}_{I,t} < \bar{x}_I^*$ for all $t < \infty$. Therefore, $\bar{x}_{I,t}$ never come back to $\bar{x}_I^*$ in finite time and thus the presumption $\bar{x}_{I,t} \leq \bar{x}_I^*$ holds with strict inequality for all $t \in (0, \infty)$.

iii) First, part ii) implies $\bar{x}_{I,t}$ reaches $\bar{x}_I^\ddagger$ in finite time as long as the last inequality $\bar{x}_I^*(1 - (1-r)r^{r/(1-r)}) \leq \bar{x}_I^\ddagger$ in the theorem is satisfied. Below we show that, for any $\bar{x}_I \leq \bar{x}_I^\ddagger$, aggregate strategy $\bar{x}_{I,t}$ cannot increase over time, i.e., $\dot{\bar{x}}_{I,t} \leq 0$ regardless of the underlying strategy composition. Hence, it cannot come back above $\bar{x}_I^\ddagger$ in any finite time; since $\bar{x}_I^\ddagger < \bar{x}_I^*$, this implies that $\bar{x}_{I,t}$ never returns to $\bar{x}_I^*$ even asymptotically. Since Theorem 5 guarantees convergence of $\bar{x}_{I,t}$ to either one aggregate equilibrium, it must converge to another aggregate equilibrium $\bar{x}_I = 0$.

Now we prove $\dot{\bar{x}}_I \leq 0$ whenever $\bar{x}_I \leq \bar{x}_I^\ddagger$. By the definition of $\bar{x}_I^\ddagger$, we have $P_\Theta^{-1}(\bar{x}_I) - F_I^0(\bar{x}_I) \geq F_I^0(\bar{x}_I) - \underline{\theta}_O$. The additional assumption on the switching rate function implies

$$\begin{aligned} R(\theta_O - F_I(\bar{x}_I)) &\geq R(P_\Theta^{-1}(\bar{x}_I) - F_I^0(\bar{x}_I) \\ &\geq R(F_I^0(\bar{x}_I) - \underline{\theta}_O) \geq R(F_I^0(\bar{x}_I) - \theta'_O) \end{aligned}$$

for any types $\theta_O \geq P_\Theta^{-1}(\bar{x}_I)$ and $\theta'_O \leq F_I(\bar{x}_I)$. The best response is I for the latter types except the exact type $\theta'_O = F_I(\bar{x}_I)$ and O for all the other types, including the former types.

Therefore, the transition of Bayesian strategy x_I is determined as follows. For any type $\theta'_O < F_I(\bar{x}_I)$, we have

$$\dot{x}_I(\theta'_O) = R(F_I^0(\bar{x}_I) - \theta'_O) x_O(\theta'_O) \leq R(F_I^0(\bar{x}_I) - \underline{\theta}_O) x_O(\theta'_O).$$

For the exact type $\theta'_O = F_I(\bar{x}_I)$, we have $\dot{x}_I(F_I(\bar{x}_I)) = 0$ by $R(F_I(\bar{x}_I) - \theta'_O) = 0$. These imply

$$\dot{X}_I(\{\theta_O : \theta_O \leq F_I(\bar{x}_I)\}) \leq R(F_I^0(\bar{x}_I) - \underline{\theta}_O) X_O(\{\theta_O : \theta_O \leq F_I(\bar{x}_I)\}).$$

For any other type $\theta_O > F_I(\bar{x}_I)$, we have

$$\dot{x}_I(\theta_O) = -R(\theta_O - F_I(\bar{x}_I)) x_I(\theta'_O) \leq 0.$$

In particular, for any type $\theta_O > P_\Theta^{-1}(\bar{x}_I)$, it is the case that

$$\dot{x}_I(\theta_O) = -R(\theta_O - F_I(\bar{x}_I)) x_I(\theta'_O) \leq -R(F_I^0(\bar{x}_I) - \underline{\theta}_O) x_I(\theta'_O).$$

Hence, we obtain

$$\dot{X}_I(\{\theta_O : \theta_O > P_\Theta^{-1}(\bar{x}_I)\}) \leq -R(F_I^0(\bar{x}_I) - \underline{\theta}_O) X_I(\{\theta_O : \theta_O > P_\Theta^{-1}(\bar{x}_I)\})$$

and

$$\dot{X}_I(\{\theta_O : \theta_O \in (F_I(\bar{x}_I), P_\Theta^{-1}(\bar{x}_I)]\}) \leq 0.$$

Note that we have

$$\begin{aligned}
& X_I(\{\theta_O : \theta_O > P_\Theta^{-1}(\bar{x}_I)\}) = (1 - P_\Theta(P_\Theta^{-1}(\bar{x}_I))) - X_O(\{\theta_O : \theta_O > P_\Theta^{-1}(\bar{x}_I)\}) \\
& \quad = 1 - \bar{x}_I - X_O(\{\theta_O : \theta_O > P_\Theta^{-1}(\bar{x}_I)\}) = \bar{x}_O - X_O(\{\theta_O : \theta_O > P_\Theta^{-1}(\bar{x}_I)\}) \\
\therefore \quad & X_I(\{\theta_O : \theta_O > P_\Theta^{-1}(\bar{x}_I)\}) - X_O(\{\theta_O : \theta_O \leq F_I(\bar{x}_I)\}) \\
& \quad = \bar{x}_O - X_O(\{\theta_O : \theta_O > P_\Theta^{-1}(\bar{x}_I)\}) - X_O(\{\theta_O : \theta_O \leq F_I(\bar{x}_I)\}) \\
& \quad = X_O(\{\theta_O : \theta_O \in (F_I(\bar{x}_I), P_\Theta^{-1}(\bar{x}_I)]\}).
\end{aligned}$$

Therefore, these changes in strategy composition add to

$$\begin{aligned}
\dot{\bar{x}}_I &= \dot{X}_I(\{\theta_O : \theta_O \leq F_I(\bar{x}_I)\}) + \dot{X}_I(\{\theta_O : \theta_O \in (F_I(\bar{x}_I), P_\Theta^{-1}(\bar{x}_I)]\}) + \dot{X}_I(\{\theta_O : \theta_O > P_\Theta^{-1}(\bar{x}_I)\}) \\
&\leq -R(F_I^0(\bar{x}_I) - \underline{\theta}_O) \left\{ X_I(\{\theta_O : \theta_O > P_\Theta^{-1}(\bar{x}_I)\}) - X_O(\{\theta_O : \theta_O \leq F_I(\bar{x}_I)\}) \right\} \\
&= -R(F_I^0(\bar{x}_I) - \underline{\theta}_O) X_O(\{\theta_O : \theta_O \in (F_I(\bar{x}_I), P_\Theta^{-1}(\bar{x}_I)]\}) \leq 0.
\end{aligned}$$

□

C Appendix to Section 5

For stability, we use the weak topology and apply the Lyapunov stability theorem, as in Cheung (2014). See its Section 4 for the detailed explanation on the strong and weak topology in evolutionary dynamics in continuous space.

Theorem 10 (Cheung, 2014: Theorems 5–6, Corollary 2)**.** *Let $Z \subset \mathcal{X}$ be a closed set and let $Y \subset \mathcal{X}$ be a neighborhood of Z in the weak topology on $\mathcal{X}$. Let $\mathcal{L} : Y \to \mathbb{R}$ be a decreasing Lyapunov function for dynamic $\mathbf{V}$: that is, $\mathcal{L}$ is continuous with respect to the weak topology and Fréchet-differentiable with $\dot{\mathcal{L}}(\mathbf{X}) = \langle \nabla \mathcal{L}(\mathbf{X}), \mathbf{V}[\mathbf{X}] \rangle \leq 0$ for all $\mathbf{X} \in Y$. Then the following holds.*

i) Any solution path starting from Y converges to the set $\{\mathbf{X} \in Y \mid \dot{\mathcal{L}}(\mathbf{X}) = 0\}$ with respect to the weak topology; i.e., this set is attracting under $\mathbf{V}$.

ii) If $\mathcal{L}^{-1}(0) = Z$, Z is Lyapunov stable under $\mathbf{V}$ with respect to the weak topology. Furthermore, if $\dot{\mathcal{L}}(\mathbf{X}) < 0$ whenever $\mathbf{X} \in Y \setminus Z$, then Z is asymptotically stable under $\mathbf{V}$.

Part i) holds for an increasing Lyapunov function; part ii) is retained by defining Z as an isolated set of local maxima.

C.1 Proof of Theorem 4

Proof. $\mathbf{x}$ being a Bayesian equilibrium is equivalent to $\mathbf{x}(\boldsymbol{\theta}) \in B[\mathbb{E}_\Theta \mathbf{x}](\boldsymbol{\theta})$ for $\mathbb{P}_\Theta$-almost all types $\boldsymbol{\theta}$. Then, for such $\boldsymbol{\theta}$, this is equivalent to $\mathbf{v}^{\mathbf{F}}[\mathbf{x}](\boldsymbol{\theta}) = \mathbf{0}$ by (6). It holds for $\mathbb{P}_\Theta$-almost all types $\boldsymbol{\theta}$, which means stationarity of Bayesian strategy $\mathbf{x}$. Note that $\mathbf{x}$ being a Bayesian equilibrium is equivalent to the corresponding strategy composition $\mathbf{X} = \int \mathbf{x} d\mathbb{P}_\Theta$ being an equilibrium composition and that stationarity of Bayesian strategy $\mathbf{x}$ is equivalent to stationarity of strategy composition $\mathbf{X}$, i.e., $\mathbf{V}^{\mathbf{F}}[\mathbf{X}] = \mathbf{O}$.[49] □

C.2 Proof of Theorem 6

In terms of strategy composition $\mathbf{X} = \int \mathbf{x} d\mathbb{P}_\Theta \in \mathcal{X}$, we redefine the heterogeneous potential function $f : \mathcal{X} \to \mathbb{R}$ as

$$f(\mathbf{X}) = f^0(\mathbf{X}(\Theta)) + \int_\Theta \sum_{a \in \mathcal{A}} \theta_a X_a(d\boldsymbol{\theta}).$$

Proof. As $f(\mathbf{X}) = f^0(\mathbf{X}(\Theta)) + \mathbb{E}_\Theta[\boldsymbol{\theta} \cdot \mathbf{x}(\boldsymbol{\theta})]$, weak continuity of f is obtained from continuity of f^0 and the dominated convergence theorem. By applying the definition of f to $\mathbf{X} + \Delta\mathbf{X}$, we have

$$\begin{aligned}
f(\mathbf{X} + \Delta\mathbf{X}) &= f^0(\bar{\mathbf{x}} + \Delta\bar{\mathbf{x}}) + \int_\Theta \sum_{a \in \mathcal{A}} \theta_a (X_a + \Delta X_a)(d\boldsymbol{\theta}) \\
&= \left\{ f^0(\bar{\mathbf{x}}) + \nabla f^0(\bar{\mathbf{x}}) \cdot \Delta\bar{\mathbf{x}} + o(|\Delta\bar{\mathbf{x}}|) \right\} + \left\{ \int_\Theta \sum_{a \in \mathcal{A}} \theta_a X_a(d\boldsymbol{\theta}) + \int_\Theta \sum_{a \in \mathcal{A}} \theta_a \Delta X_a(d\boldsymbol{\theta}) \right\} \\
&= f(\mathbf{X}) + \mathbf{F}^0(\bar{\mathbf{x}}) \cdot \Delta\bar{\mathbf{x}} + \int_\Theta \sum_{a \in \mathcal{A}} \theta_a \Delta X_a(d\boldsymbol{\theta}) + o(|\Delta\bar{\mathbf{x}}|) \\
&= f(\mathbf{X}) + \int_\Theta \sum_{a \in \mathcal{A}} (F_a^0(\bar{\mathbf{x}}) + \theta_a) \Delta X_a(d\boldsymbol{\theta}) + o(|\Delta\bar{\mathbf{x}}|).
\end{aligned}$$

Here $\bar{\mathbf{x}} = \mathbf{X}(\Theta)$ and $\Delta\bar{\mathbf{x}} = \Delta\mathbf{X}(\Theta)$. The second equality comes from differentiability of f^0; the third is from the assumption that f^0 is a potential function of $\mathbf{F}^0$ and the definition of f applied to $\mathbf{X}$. Then, we should recall $F_a[\mathbf{X}(\Theta)](\boldsymbol{\theta}) = F_a^0(\mathbf{X}(\Theta)) + \theta_a$. So the second term is $\langle \mathbf{F}[\mathbf{X}(\Theta)], \Delta\mathbf{X} \rangle$. About the third error term, note that $|\Delta\bar{\mathbf{x}}| = |\Delta\mathbf{X}(\Theta)| \leq \|\Delta\mathbf{X}\|$. Therefore, we obtain

$$f(\mathbf{X} + \Delta\mathbf{X}) = f(\mathbf{X}) + \langle \mathbf{F}[\mathbf{X}(\Theta)], \Delta\mathbf{X} \rangle + o(\|\Delta\mathbf{X}\|).$$

Thus, f is (Fréchet) differentiable with derivative $\nabla f(\mathbf{X}) \equiv \mathbf{F}[\mathbf{X}(\Theta)]$. So we have verified that f is a potential function of the game $\mathbf{F}$ defined on $\mathcal{X}$. □

C.3 Proof of Theorem 5

Proof. **i)** Since f is a potential function for $\mathbf{F}$, we have

$$\dot{f}(\mathbf{X}) = \langle \nabla f(\mathbf{X}), \dot{\mathbf{X}} \rangle = \langle \mathbf{F}[\bar{\mathbf{x}}], \mathbf{V}^{\mathbf{F}}[\mathbf{X}] \rangle = \mathbb{E}_\Theta \left[\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta}) \cdot \mathbf{v}^{\mathbf{F}}[\mathbf{x}](\boldsymbol{\theta}) \right],$$

where $\bar{\mathbf{x}} = \mathbf{X}(\Theta)$ and $\mathbf{X} = \int_\Theta \mathbf{x} d\mathbb{P}_\Theta$.

Since $\mathbf{v}^{\mathbf{F}}[\mathbf{x}](\boldsymbol{\theta}) = \mathbf{v}(\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta}), \mathbf{x}(\boldsymbol{\theta}))$, the first part of (8) implies $\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta}) \cdot \mathbf{v}^{\mathbf{F}}[\mathbf{x}](\boldsymbol{\theta}) \geq 0$ for all $\boldsymbol{\theta}$ and thus

$$\dot{f}(\mathbf{X}) = \mathbb{E}_\Theta \left[\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta}) \cdot \mathbf{v}^{\mathbf{F}}[\mathbf{x}](\boldsymbol{\theta}) \right] \geq 0.$$

[49] $\mathbf{O} = (O_a)_{a \in \mathcal{A}} \in \mathcal{M}_{\mathcal{A} \times \Theta}$ denotes a zero vector measure such as $O_a(B_\Theta) = 0$ for any $B_\Theta \in \mathcal{B}_\Theta, a \in \mathcal{A}$.

Suppose $\mathbf{x}$ is not a Bayesian equilibrium. By Theorem 4, this is equivalent to non-stationarity of the Bayesian strategy $\mathbf{x}$, i.e., $\mathbb{P}_\Theta\{\boldsymbol{\theta} : \mathbf{v}^{\mathbf{F}}[\mathbf{x}](\boldsymbol{\theta}) \neq \mathbf{0}\}) > 0$. For a type with $\mathbf{v}^{\mathbf{F}}[\mathbf{x}](\boldsymbol{\theta}) \neq \mathbf{0}$, the second part of (8) implies $\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta}) \cdot \mathbf{v}^{\mathbf{F}}[\mathbf{x}](\boldsymbol{\theta}) > 0$. Since this holds for positive mass of types, we have

$$\dot{f}(\mathbf{X}) = \mathbb{E}_\Theta\left[\mathbf{F}[\bar{\mathbf{x}}](\boldsymbol{\theta}) \cdot \mathbf{v}^{\mathbf{F}}[\mathbf{x}](\boldsymbol{\theta})\right] > 0.$$

Therefore, f is a strictly increasing Lyapunov function. By Theorem 10, this implies that the set of Bayesian equilibria is globally attracting; a local strict maximum of f is locally asymptotically stable.

ii) Suppose that the corresponding isolated equilibrium composition $\mathbf{X}^*$ is asymptotically stable, with the basin of attraction $\mathcal{X}^* \subset \mathcal{X}$. Take an arbitrary strategy composition $\mathbf{X}_0$ from $\mathcal{X}^*$ and let $\{\mathbf{X}_t\}_{t\in\mathbb{R}_+}$ be the solution trajectory of the heterogeneous dynamic from $\mathbf{X}_0$. Since f is a strictly increasing Lyapunov function, it must be the case that $\dot{f}(\mathbf{X}_t) > 0$ as long as $\mathbf{X}_t$ has not reached exactly $\mathbf{X}^*$. Thus, $f(\mathbf{X}^*) = f(\mathbf{X}_0) + \int_0^\infty \dot{f}(\mathbf{X}_t)dt > f(\mathbf{X}_0)$. Since $\mathbf{X}_0$ is taken arbitrarily from $\mathcal{X}^*$, this verifies that $\mathbf{X}^*$ maximizes f in this neighborhood $\mathcal{X}^*$. □

C.4 Proof of Theorem 7

In terms of strategy compositions, the first part of the lemma is rephrased as $\bar{f}(\mathbf{X}(\Theta)) \geq f(\mathbf{X})$ for any $\mathbf{X} \in \mathcal{X}$; the second part means that the equality in this equation is equivalent to $\mathbf{X}$ being an equilibrium composition.

Proof. **i)** From (9), observe that, for any $\mathbf{X} \in \mathcal{X}$ and $\bar{\boldsymbol{\pi}} \in \mathbb{R}^A$

$$\begin{aligned} f(\mathbf{X}) &= f^0(\mathbf{X}(\Theta)) + \int_\Theta \sum_{a\in\mathcal{A}} (\bar{\pi}_a + \theta_a) x_a(\boldsymbol{\theta}) \mathbb{P}(d\boldsymbol{\theta}) - \bar{\boldsymbol{\pi}} \cdot \mathbf{X}(\Theta) \\ &\leq f^0(\mathbf{X}(\Theta)) + \mathbb{E}_\Theta[\max_{i\in\mathcal{A}}(\bar{\pi}_i + \theta_i)] - \bar{\boldsymbol{\pi}} \cdot \mathbf{X}(\Theta). \end{aligned} \tag{C.6}$$

As this holds for any $\bar{\boldsymbol{\pi}} \in \mathbb{R}^A$, the definition of $\bar{f}$ implies $f(\mathbf{X}) \leq \bar{f}(\mathbf{X}(\Theta))$.

ii) To find the condition for equality, notice that the inequality holds in (C.6) with equality if and only if $\mathbf{x}(\boldsymbol{\theta}) \in \operatorname{argmax}_{\mathbf{y}\in\Delta^A} \mathbf{y} \cdot (\bar{\boldsymbol{\pi}} + \boldsymbol{\theta})$ for $\mathbb{P}_\Theta$-almost all $\boldsymbol{\theta} \in \Theta$; that is, in composition $\mathbf{X}$, almost all agents take optimal action given $\bar{\boldsymbol{\pi}}$. If $\bar{\boldsymbol{\pi}} = \mathbf{F}^0(\bar{\mathbf{x}})$, then it means $\mathbf{X}$ is an equilibrium composition with $\bar{\mathbf{x}} = \mathbf{X}(\Theta)$.

According to Hofbauer and Sandholm (2007), the minimized function in (10) is strictly convex and has the partial derivative w.r.t. $\bar{x}_a$ equal to $\mathbb{P}_\Theta(a = \arg\max_{i\in\mathcal{A}}(\bar{\pi}_i + \theta_i))$. So

$$\bar{f}(\mathbf{X}(\Theta)) = f^0(\mathbf{X}(\Theta)) + \mathbb{E}_\Theta[\max_{i\in\mathcal{A}}(\bar{\pi}_i + \theta_i)] - \bar{\boldsymbol{\pi}} \cdot \mathbf{X}(\Theta)$$

if and only if $\bar{\boldsymbol{\pi}}$ satisfies $\bar{x}_a = \mathbb{P}_\Theta(a = \arg\max_{i\in\mathcal{A}}(\bar{\pi}_i + \theta_i))$ for each $a \in \mathcal{A}$. If $\bar{\mathbf{x}}^*$ is an aggregate equilibrium, then $\bar{x}^*_a = \mathbb{P}_\Theta(a = \arg\max_{i\in\mathcal{A}}(F^0_i(\bar{\mathbf{x}}^*) + \theta_i))$ and thus

$$\bar{f}(\bar{\mathbf{x}}^*) = f^0(\bar{\mathbf{x}}^*) + \mathbb{E}_\Theta[\max_{i\in\mathcal{A}}(F^0_i(\bar{\mathbf{x}}^*) + \theta_i)] - \mathbf{F}^0(\bar{\mathbf{x}}^*) \cdot \bar{\mathbf{x}}^*.$$

Therefore, if and only if $\mathbf{X}^*$ is an equilibrium composition, we have

$$f(\mathbf{X}^*) = f^0(\mathbf{X}^*(\Theta)) + \mathbb{E}_\Theta[\max_{i \in \mathcal{A}}(F_i^0(\mathbf{X}^*(\Theta)) + \theta_i)] - \mathbf{F}^0(\mathbf{X}^*(\Theta)) \cdot \mathbf{X}^*(\Theta) = \bar{f}(\mathbf{X}^*(\Theta)).$$

Otherwise, we have $f(\mathbf{X}) < \bar{f}(\mathbf{X}(\Theta))$.

iii) This is immediate from Corollary 2 and the fact that $\bar{f}$ is a strictly increasing Lyapunov function for the homogenized smooth BRD (Hofbauer and Sandholm, 2007, Theorem 3.2). □

Supplementary note

"Nonaggregable evolutionary dynamics under payoff heterogeneity"

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October 8, 2017

S1 Supplementary note on Section 1

S1.1 Example in Figure 1

The transition vector at time 0

At the initial aggregate strategy $\bar{\mathbf{x}}_0 = 0.5\bar{\mathbf{x}}_0(\theta^H) + 0.5\bar{\mathbf{x}}_0(\theta^L) = (\varepsilon, 0.5(1-\varepsilon), 0.5(1-\varepsilon))$, the payoff vector for type θ is $\mathbf{F}[\bar{\mathbf{x}}_0](\theta) = (1+\theta, 1, 1)$. Therefore, as long as $\theta > 0$, action A is the unique best response and the other two actions B and C are equally worse than A. Therefore, in all the major dynamics illustrated in the subfigures, all agents switch to A and none switch between B and C. Besides, since action B and action C initially yield the same payoffs and the aggregate masses of these two actions are initially equal, the switching rate from B to A and that from C to A are equal to each other at time 0, though they depend on the type of agents and they will differ at later periods of time as the symmetry in the aggregate strategy will break. Given the initial strategy composition $\mathbf{x}_0$, none of type-θ^H agents take action B at time 0 and none of type-θ^L agents take action C. But this symmetry in the switching rate is important below, when comparing the transition from this strategy composition and that from another flipped composition.

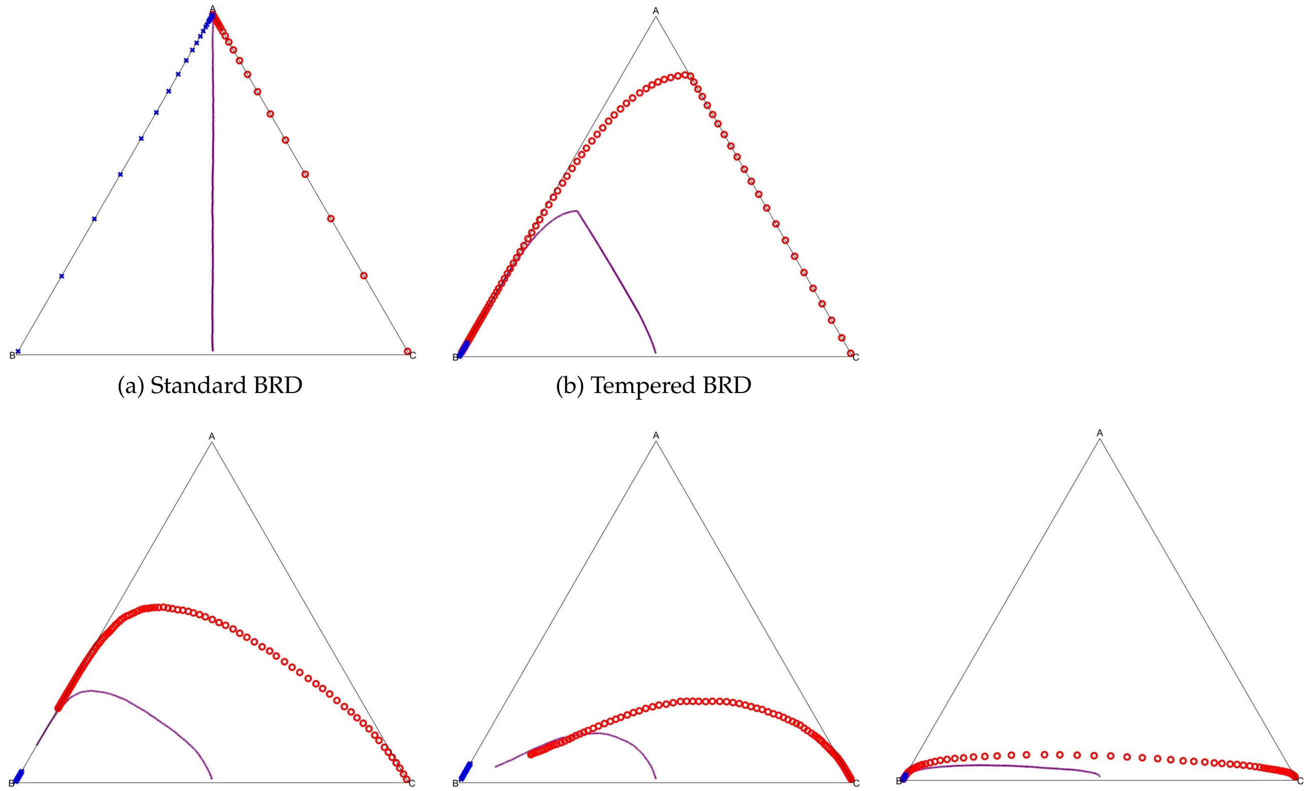
Denote this initial switching rate for type θ^H by R_0^H and that for type θ^L by R_0^L. These are obtained for each dynamic as in Table 1.

Therefore, the transition vectors of each type's Bayesian strategy at time 0 are obtained as follows:

$$\dot{\mathbf{x}}_0(\theta^H) = R_0^H(1-\varepsilon)(\mathbf{e}^A - \mathbf{e}^B),$$
$$\dot{\mathbf{x}}_0(\theta^L) = R_0^L(1-\varepsilon)(\mathbf{e}^A - \mathbf{e}^C).$$

Dynamic	R_0^H	R_0^L
Standard BRD	1	1
Tempered BRD	$Q(\theta^H)$	$Q(\theta^L)$
Smith	θ^H	θ^L
BNN	$(1-\varepsilon)\theta^H$	$(1-\varepsilon)\theta^L$
Replicator	$\varepsilon\theta^H$	$\varepsilon\theta^L$

Table 1: Switching rate from a suboptimal action to the optimal action A, given the strategy composition $\bar{\mathbf{x}}_0$ as in Figure 1. For BNN, it is assumed that a revising agent compares his own payoff with the average payoff of agents of the same type. For replicator, since the proportion of action-A players is ε both in the aggregate strategy or in each type's Bayesian strategy, it does not matter whether agents are sampling from the whole population or the same type of agents.



(a) Standard BRD (b) Tempered BRD (c) Smith (d) BNN (e) Replicator

Figure 5: Dynamics of aggregate strategy in a symmetric 3-action coordination game from $\mathbf{x}_0'$. The notation of markers is the same as Figure 2.

By aggregating these transition vectors over the two types, we can get the transition vector of the aggregate strategy at time 0:

$$\begin{aligned}\dot{\bar{\mathbf{x}}}_0 &= 0.5\dot{\mathbf{x}}_0(\theta^H) + 0.5\dot{\mathbf{x}}_0(\theta^L) \\ &= (1-\varepsilon)\left\{R_0^H(\mathbf{e}^A - \mathbf{e}^B) + R_0^L(\mathbf{e}^A - \mathbf{e}^C)\right\}.\end{aligned}$$

Thus, the initial transition vector is asymmetrically tilted toward $\mathbf{e}^A - \mathbf{e}^B$, if $R_0^H > R_0^L$. From Table 1, we can see that this is the case for all the dynamics, except the standard BRD.

Comparison with another initial strategy composition

To confirm the dependency of the aggregate strategy trajectories on the initial strategy composition, we consider another strategy composition $\mathbf{x}_0'$:

$$\mathbf{x}_0'(\theta^H) = (\varepsilon, 0, 1-\varepsilon), \qquad \mathbf{x}_0'(\theta^H) = (\varepsilon, 1-\varepsilon, 0).$$

This shares the same aggregate strategy $\bar{\mathbf{x}}_0' = 0.5\bar{\mathbf{x}}_0'(\theta^H) + 0.5\bar{\mathbf{x}}_0'(\theta^L) = (\varepsilon, 0.5(1-\varepsilon), 0.5(1-\varepsilon))$ as the first example. Figure 5 shows the trajectory of aggregate strategy under each of the five major dynamics, starting from $\mathbf{x}_0'$. Note that, by the same calculation as above, we can easily obtain the transition vector of the aggregate strategy at time 0 as

$$\dot{\mathbf{x}}_0' = (1-\varepsilon)\left\{R_0^H(\mathbf{e}^A - \mathbf{e}^C) + R_0^L(\mathbf{e}^A - \mathbf{e}^B)\right\}.$$

Note that the switching rate of each type from a suboptimal action to the optimal action A is the same as in the last example, i.e., the same as in Table 1. Therefore, if $R_0^H > R_0^L$, i.e., in all the dynamics exthe initial transition vector is now tilted toward $\mathbf{e}^A - \mathbf{e}^C$.

S2 Supplementary note on Sections 2–3

S2.1 Norms on $\mathcal{X}$ and on $\mathcal{F}_{\mathcal{X}}$

Proof of Theorem 8

Proof. An arbitrary measurable function $\mathbf{g} : \Theta \to \mathbb{R}^A$ bounded by 1 satisfies

$$\sum_{a\in\mathcal{A}} \int_{\boldsymbol{\theta}\in\Theta} g_a(\boldsymbol{\theta}) M_a(d\boldsymbol{\theta}) = \sum_{a\in\mathcal{A}} \int_{\boldsymbol{\theta}\in\Theta} g_a(\boldsymbol{\theta}) m_a(\boldsymbol{\theta}) \mathbb{P}_\Theta(d\boldsymbol{\theta})$$

$$\therefore \left| \sum_{s\in\mathcal{A}} \int_{\boldsymbol{\theta}\in\Theta} g_a(\boldsymbol{\theta}) m_a(\boldsymbol{\theta}) \mathbb{P}_\Theta(d\boldsymbol{\theta}) \right| \le \sum_{a\in\mathcal{A}} \int_{\boldsymbol{\theta}\in\Theta} |g_a(\boldsymbol{\theta}) m_a(\boldsymbol{\theta})| \mathbb{P}_\Theta(d\boldsymbol{\theta}) \le \sum_{a\in\mathcal{A}} \int_{\boldsymbol{\theta}\in\Theta} |m_a(\boldsymbol{\theta})| \mathbb{P}_\Theta(d\boldsymbol{\theta})$$

The last inequality comes from g_a being bounded by 1. As this holds for any such $\mathbf{g}$, the supremum cannot exceed $\sum_a \int_\Theta |m_a| d\mathbb{P}_\Theta$.

On the other hand, define function $\bar{\mathbf{g}} : \Theta \to \mathbb{R}^A$ by $\bar{g}_a(\boldsymbol{\theta}) = \mathbb{1}\{m_a(\boldsymbol{\theta}) > 0\} - \mathbb{1}\{m_a(\boldsymbol{\theta}) \le 0\}$. Then,

$$\begin{aligned}
\|\mathbf{M}\| &\ge \sum_{a\in\mathcal{A}} \int_{\boldsymbol{\theta}\in\Theta} \bar{g}_a(\boldsymbol{\theta}) M_a(d\boldsymbol{\theta}) \\
&= \sum_{a\in\mathcal{A}} \int_{\boldsymbol{\theta}\in m_a^{-1}(\mathbb{R}_{++})} 1 \cdot m_a(\boldsymbol{\theta}) \mathbb{P}_\Theta(d\boldsymbol{\theta}) + \int_{\boldsymbol{\theta}\in m_a^{-1}(\mathbb{R}_{-})} (-1) \cdot m_a(\boldsymbol{\theta}) \mathbb{P}_\Theta(d\boldsymbol{\theta}) \\
&= \sum_{a\in\mathcal{A}} \int_{\boldsymbol{\theta}\in m_a^{-1}(\mathbb{R}_{++})} |m_a(\boldsymbol{\theta})| \mathbb{P}_\Theta(d\boldsymbol{\theta}) + \int_{\boldsymbol{\theta}\in m_a^{-1}(\mathbb{R}_{-})} |m_a(\boldsymbol{\theta})| \mathbb{P}_\Theta(d\boldsymbol{\theta}) \\
&= \sum_{a\in\mathcal{A}} \int_{\boldsymbol{\theta}\in\Theta} |m_a(\boldsymbol{\theta})| \mathbb{P}_\Theta(d\boldsymbol{\theta}).
\end{aligned}$$

Combining these two inequalities, we verify the claim. □

S2.2 Proof of Theorem 1

Lebesgue decomposition

Lemma 1 (cf. Rudin, 1987: §6.10)**.** *For any finite signed measure $\mathbf{M} = (M_a)_{a\in\mathcal{A}} \in \mathcal{M}_{\mathcal{A}\times\Theta}$, there is a pair of finite signed measures $\tilde{\mathbf{M}} = (\tilde{M}_a)_{a\in\mathcal{A}}, \hat{\mathbf{M}} = (\hat{M}_a)_{a\in\mathcal{A}} \in \mathcal{M}_{\mathcal{A}\times\Theta}$ such that, for each $a \in \mathcal{A}$,*

i) $M_a = \tilde{M}_a + \hat{M}_a$*;*

ii) $\tilde{M}_a \ll \mathbb{P}_\Theta$*, i.e.,* $\mathbb{P}_\Theta(B_\Theta) = 0 \implies \tilde{M}_a(B_\Theta) = 0$ *for any* $B_\Theta \in \mathcal{B}_\Theta$

iii) $\hat{M}_a \perp \mathbb{P}_\Theta$*, i.e., there exists* $E_a \in \mathcal{B}_\Theta$ *such that* $\hat{M}_a(B_\Theta \cap E_a) = 0$ *and* $\mathbb{P}_\Theta(B_\Theta \setminus E_a) = 0$ *for any* $B_\Theta \in \mathcal{B}_\Theta$*.*

The part (ii) implies that $\tilde{\mathbf{M}}$ has density $\tilde{\mathbf{m}} = (\tilde{m}_a)_{a\in\mathcal{A}}$ with respect to $\mathbb{P}_\Theta$. Besides, $\|\tilde{\mathbf{M}}\| \le \|\mathbf{M}\|$, since i) and ii) imply $\|\mathbf{M}\| = \|\tilde{\mathbf{M}}\| + \|\hat{\mathbf{M}}\|$.

Proof of Lipschitz continuity of V (part i of Theorem 1)

Henceforth, as we focus on $\mathbb{P}_\Theta$-absolute continuous finite signed measures, we omit the tilde from such measures.

In the following proofs of part i for the two kinds of protocols, we consider two $\mathbb{P}_\Theta$-absolute continuous finite signed measures $\mathbf{M}, \mathbf{M}' \in \tilde{\mathcal{M}}_{\mathcal{A}\times\Theta}$ with densities $\mathbf{m}$ and $\mathbf{m}'$. Let $\bar{\mathbf{m}} = \mathbf{M}(\Theta)$, $\bar{\boldsymbol{\mu}} = \boldsymbol{\mu}(\bar{\mathbf{m}})$, $\boldsymbol{\mu}(\boldsymbol{\theta}) := \boldsymbol{\mu}(\mathbf{m}(\boldsymbol{\theta}))$ and $R^{\mathbf{F}}_{ji}(\boldsymbol{\theta}) := R_{ji}(\mathbf{F}[\bar{\boldsymbol{\mu}}](\boldsymbol{\theta}), \boldsymbol{\mu}(\boldsymbol{\theta}))$; similarly we define $\bar{\mathbf{m}}', \bar{\boldsymbol{\mu}}', \boldsymbol{\mu}'(\boldsymbol{\theta})$ and $R'^{\mathbf{F}}_{ji}(\boldsymbol{\theta})$.

Proof of part i: continuous switching rate functions. Let $L_R > 0$ be the greatest Lipschitz constant of functions R_{ij} among all pairs of actions $i, j \in \mathcal{A}$. The Lipschitz continuity of $R_{\cdot\cdot}$ and $\mathbf{F}^0$ implies

$$\begin{aligned}
|R^{\mathbf{F}}_{ji}(\boldsymbol{\theta}) - R'^{\mathbf{F}}_{ji}(\boldsymbol{\theta})| &\le L_R |(\mathbf{F}[\bar{\boldsymbol{\mu}}](\boldsymbol{\theta}), \boldsymbol{\mu}(\boldsymbol{\theta})) - (\mathbf{F}[\bar{\boldsymbol{\mu}}'](\boldsymbol{\theta}), \boldsymbol{\mu}'(\boldsymbol{\theta}))| \\
&\le L_R \left\{ |\mathbf{F}[\bar{\boldsymbol{\mu}}](\boldsymbol{\theta}) - \mathbf{F}[\bar{\boldsymbol{\mu}}'](\boldsymbol{\theta})| + |\boldsymbol{\mu}(\boldsymbol{\theta}) - \boldsymbol{\mu}'(\boldsymbol{\theta})| \right\} \\
&\le L_R \left(L_{\mathbf{F}}(\boldsymbol{\theta}) |\bar{\boldsymbol{\mu}} - \bar{\boldsymbol{\mu}}'| + |\boldsymbol{\mu}(\boldsymbol{\theta}) - \boldsymbol{\mu}'(\boldsymbol{\theta})| \right) \\
&\le L_R \left(L_{\mathbf{F}}(\boldsymbol{\theta}) L_{\boldsymbol{\mu}} |\bar{\mathbf{m}} - \bar{\mathbf{m}}'| + L_{\boldsymbol{\mu}} |\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})| \right). \qquad \text{(S.1)}
\end{aligned}$$

From the definition of $v^{\mathbf{F}+}_i$, we have

$$\begin{aligned}
&|v^{\mathbf{F}+}_i[\mathbf{M}](\boldsymbol{\theta}) - v^{\mathbf{F}+}_i[\mathbf{M}'](\boldsymbol{\theta})| \le \sum_{j\in\mathcal{A}} |R^{\mathbf{F}}_{ji}(\boldsymbol{\theta}) \mu_j(\boldsymbol{\theta}) - R'^{\mathbf{F}}_{ji}(\boldsymbol{\theta}) \mu'_j(\boldsymbol{\theta})| \\
&\quad \le \sum_{j\in\mathcal{A}} \left\{ |R^{\mathbf{F}}_{ji}(\boldsymbol{\theta}) - R'^{\mathbf{F}}_{ji}(\boldsymbol{\theta})| \, |\mu_j(\boldsymbol{\theta})| + |R'^{\mathbf{F}}_{ji}(\boldsymbol{\theta})| \cdot |\mu_j(\boldsymbol{\theta}) - \mu'_j(\boldsymbol{\theta})| \right\} \\
&\quad \le \sum_{j\in\mathcal{A}} \left[3L_R \left(L_{\mathbf{F}}(\boldsymbol{\theta}) L_{\boldsymbol{\mu}} |\bar{\mathbf{m}} - \bar{\mathbf{m}}'| + L_{\boldsymbol{\mu}} |\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})| \right) + \bar{R} L_{\boldsymbol{\mu}} |\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})| \right] \\
&\quad \le A \left\{ 3L_R L_{\mathbf{F}}(\boldsymbol{\theta}) L_{\boldsymbol{\mu}} |\bar{\mathbf{m}} - \bar{\mathbf{m}}'| + (3L_R + \bar{R}) L_{\boldsymbol{\mu}} |\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})| \right\} \qquad \text{(S.2)}
\end{aligned}$$

Here the third inequality comes from (S.1), Assumption 2 and $|\mu_j(\cdot)| \le 3$. Similarly, we get

$$|v^{\mathbf{F}-}_i[\mathbf{M}](\boldsymbol{\theta}) - v^{\mathbf{F}-}_i[\mathbf{M}'](\boldsymbol{\theta})| \le A \left\{ 3L_R L_{\mathbf{F}} L_{\boldsymbol{\mu}} |\bar{\mathbf{m}} - \bar{\mathbf{m}}'| + (3L_R + \bar{R}) L_{\boldsymbol{\mu}} |\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})| \right\}.$$

Therefore, we have

$$\begin{aligned}
&\|\mathbf{V}^{\mathbf{F}}[\mathbf{M}] - \mathbf{V}^{\mathbf{F}}[\mathbf{M}']\| \\
&\quad \le \int_\Theta \sum_{i\in\mathcal{A}} \left(|v^{\mathbf{F}+}_i[\mathbf{M}](\boldsymbol{\theta}) - v^{\mathbf{F}+}_i[\mathbf{M}'](\boldsymbol{\theta})| + |v^{\mathbf{F}-}_i[\mathbf{M}](\boldsymbol{\theta}) - v^{\mathbf{F}-}_i[\mathbf{M}'](\boldsymbol{\theta})| \right) \mathbb{P}_\Theta(d\boldsymbol{\theta}) \\
&\quad \le \int_\Theta \left[\sum_{i\in\mathcal{A}} 2A \left\{ 3L_R L_{\mathbf{F}}(\boldsymbol{\theta}) L_{\boldsymbol{\mu}} |\bar{\mathbf{m}} - \bar{\mathbf{m}}'| + (3L_R + \bar{R}) L_{\boldsymbol{\mu}} |\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})| \right\} \right] \mathbb{P}_\Theta(d\boldsymbol{\theta}) \\
&\quad = 2A^2 \cdot 3L_R \mathbb{E}_\Theta L_{\mathbf{F}} L_{\boldsymbol{\mu}} |\bar{\mathbf{m}} - \bar{\mathbf{m}}'| + 2A^2 (3L_R + \bar{R}) L_{\boldsymbol{\mu}} \int_\Theta |\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})| \mathbb{P}_\Theta(d\boldsymbol{\theta})
\end{aligned}$$

$$\leq 2A^2(3L_R\bar{L}_{\mathbf{F}} + 3L_R + \bar{R})L_{\boldsymbol{\mu}}\|\mathbf{M} - \mathbf{M}'\|.$$

The last inequality comes from $|\bar{\mathbf{m}} - \bar{\mathbf{m}}'| = |\mathbf{M}(\Theta) - \mathbf{M}'(\Theta)| \leq \|\mathbf{M} - \mathbf{M}'\|$. So $\mathbf{V}^{\mathbf{F}}$ is Lipschitz continuous with constant $2A^2(3L_R\bar{L}_{\mathbf{F}} + 3L_R + \bar{R})L_{\boldsymbol{\mu}}$. □

In an exact optimization protocol, the dynamic reduces as the following: if b is the unique maximizer of $\{\tilde{F}_a(\bar{\boldsymbol{\mu}};\boldsymbol{\theta}) \mid a \in \mathcal{A}\}$, i.e., the unique best response to $\bar{\boldsymbol{\mu}}$ for type $\boldsymbol{\theta}$, then

$$v_b(\boldsymbol{\theta})[\mathbf{M}] = \sum_{j\in\mathcal{A}\setminus\{b\}} Q_{jb}(\tilde{\mathbf{F}}(\bar{\boldsymbol{\mu}};\boldsymbol{\theta}))\mu_j(\boldsymbol{\theta})$$
$$v_i(\boldsymbol{\theta})[\mathbf{M}] = -Q_{ib}(\tilde{\mathbf{F}}(\bar{\boldsymbol{\mu}};\boldsymbol{\theta}))\mu_i(\boldsymbol{\theta}) \qquad \text{for any } i \in \mathcal{A}\setminus\{b\}.$$

Assumption 3 implies that the best response is unique for almost every type, this determines the composite dynamic without ambiguity.

Proof of part i: exact optimization protocols. Let $L_Q > 0$ be the greatest Lipschitz constant of functions Q_{ij} among all pairs of actions $i, j \in \mathcal{A}$. Denote $\Delta v_i(\boldsymbol{\theta}) := v_i(\boldsymbol{\theta})[\mathbf{M}] - v_i(\boldsymbol{\theta})[\mathbf{M}']$. Let $\beta_b^{-1}(\bar{\boldsymbol{\mu}})$ be the set of types $\boldsymbol{\theta}$ for whom $b \in \mathcal{A}$ is the unique optimal action given $\tilde{\mathbf{F}}(\bar{\boldsymbol{\mu}};\boldsymbol{\theta})$, and N be the set of types for which there are multiple best responses at $\tilde{\mathbf{F}}(\bar{\boldsymbol{\mu}};\boldsymbol{\theta})$ or $\tilde{\mathbf{F}}(\bar{\boldsymbol{\mu}}';\boldsymbol{\theta})$. Assumption 3 implies $\mathbb{P}_\Theta(N) = 0$. Define partitions of $\Theta \setminus N$ by

$$\cap\beta_b := \beta_b^{-1}(\bar{\mathbf{m}}) \cap \beta_b^{-1}(\bar{\boldsymbol{\mu}}'), \qquad \Delta\beta_{bc} := \beta_b^{-1}(\bar{\mathbf{m}}) \cap \beta_c^{-1}(\bar{\boldsymbol{\mu}}') \qquad \text{for each } b \in \mathcal{A}, c \in \mathcal{A}\setminus\{b\}.$$

Let $\cap\beta := \bigcup_{b\in\mathcal{A}} \cap\beta_b$ and $\Delta\beta := \bigcup_{b\in\mathcal{A}}\bigcup_{c\in\mathcal{A}\setminus\{b\}} \Delta\beta_{bc}$.

Denote $Q_{ji}(\boldsymbol{\theta}) := Q_{ji}(\tilde{\mathbf{F}}(\bar{\boldsymbol{\mu}};\boldsymbol{\theta}))$ and $Q'_{ji}(\boldsymbol{\theta}) := Q_{ji}(\tilde{\mathbf{F}}(\bar{\boldsymbol{\mu}}';\boldsymbol{\theta}))$. Similarly to (S.1), Lipschitz continuity of Q_{ji} and F^0 implies

$$|Q_{ji}(\boldsymbol{\theta}) - Q'_{ji}(\boldsymbol{\theta})| \leq L_Q L_{\boldsymbol{\mu}} \left(L_F(\boldsymbol{\theta})|\bar{\mathbf{m}} - \bar{\mathbf{m}}'| + |\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})|\right) \tag{S.3}$$

for all $i, j \in \mathcal{A}, \boldsymbol{\theta} \in \Theta$.

i) Consider $\cap\beta_b$ for an arbitrary $b \in \mathcal{A}$. Fix $\boldsymbol{\theta} \in \cap\beta_b$: action b is the optimal action for this type $\boldsymbol{\theta}$ both in the state $\mathbf{M}$ and the state $\mathbf{M}'$. Then, similarly to (S.2), boundedness of Q and (S.3) imply

$$|\Delta v_b(\boldsymbol{\theta})| \leq \sum_{j\in\mathcal{A}\setminus\{b\}} |Q_{jb}(\boldsymbol{\theta})\mu_j(\mathbf{m}(\boldsymbol{\theta})) - Q'_{jb}(\boldsymbol{\theta})\mu_j(\mathbf{m}'(\boldsymbol{\theta}))|$$
$$\leq (A-1)\left\{3L_Q L_{\mathbf{F}}(\boldsymbol{\theta})L_{\boldsymbol{\mu}}|\bar{\mathbf{m}} - \bar{\mathbf{m}}'| + (3L_Q + \bar{Q})L_{\boldsymbol{\mu}}|\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})|\right\}$$

For action $i \neq b$,

$$\Delta v_i(\boldsymbol{\theta}) = (-Q_{ib}(\boldsymbol{\theta})\mu_i(\boldsymbol{\theta})) - \left(-Q'_{ib}(\boldsymbol{\theta})\mu_i(\boldsymbol{\theta})\right) = -\{Q_{ib}(\boldsymbol{\theta}) - Q'_{ib}(\boldsymbol{\theta})\}\mu_i(\boldsymbol{\theta}) - Q'_{ib}(\boldsymbol{\theta})\{\mu_i(\boldsymbol{\theta}) - \mu'_i(\boldsymbol{\theta})\}.$$

(S.3) implies

$$|\Delta v_i(\boldsymbol{\theta})| \leq L_Q L_{\boldsymbol{\mu}}\left(L_{\mathbf{F}}(\boldsymbol{\theta})|\bar{\mathbf{m}} - \bar{\mathbf{m}}'| + |\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})|\right)|\mu_i(\boldsymbol{\theta})| + Q'_i(\boldsymbol{\theta})L_{\boldsymbol{\mu}}|\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})|$$
$$\leq 3L_Q L_{\boldsymbol{\mu}} L_{\mathbf{F}}(\boldsymbol{\theta})|\bar{\mathbf{m}} - \bar{\mathbf{m}}'| + (3L_Q + \bar{Q})L_{\boldsymbol{\mu}}|\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})|.$$

The second inequality comes from boundeness of Q and $\boldsymbol{\mu}$.

Therefore, we have

$$\sum_{a\in\mathcal{A}} |\Delta v_a(\boldsymbol{\theta})| \le 2(A-1)L_{\boldsymbol{\mu}} \left\{3L_Q L_{\mathbf{F}}(\boldsymbol{\theta})|\bar{\mathbf{m}} - \bar{\mathbf{m}}'| + (3L_Q + \bar{Q})|\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})|\right\}$$

and thus

$$\begin{aligned}
&\int_{\cap\beta} \sum_{a\in\mathcal{A}} |\Delta v_a(\boldsymbol{\theta})|\mathbb{P}_\Theta(d\boldsymbol{\theta}) \\
&\le 2(A-1)L_{\boldsymbol{\mu}} \left\{3L_Q \int_{\cap\beta} L_{\mathbf{F}}(\boldsymbol{\theta})|\bar{\mathbf{m}} - \bar{\mathbf{m}}'|\mathbb{P}_\Theta(d\boldsymbol{\theta}) + (3L_Q + \bar{Q}) \int_{\cap\beta} |\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})|\mathbb{P}_\Theta(d\boldsymbol{\theta})\right\} \\
&\le 2(A-1)L_{\boldsymbol{\mu}}(3L_Q\bar{L}_{\mathbf{F}} + 3L_Q + \bar{Q})\|\mathbf{M} - \mathbf{M}'\|. \qquad \text{(S.4)}
\end{aligned}$$

The second inequality comes from $\mathbb{P}_\Theta(\cap\beta) \le \mathbb{P}_\Theta(\Theta) = 1$, $|\bar{\mathbf{m}} - \bar{\mathbf{m}}'| \le \|\mathbf{M} - \mathbf{M}'\|$, $\int_{\cap\beta} L_{\mathbf{F}}(\boldsymbol{\theta})\mathbb{P}_\Theta(d\boldsymbol{\theta}) \le \mathbb{E}_\Theta L_{\mathbf{F}} = \bar{L}_{\mathbf{F}}$, and $\int_{\cap\beta} |\mathbf{m}(\boldsymbol{\theta}) - \mathbf{m}'(\boldsymbol{\theta})|\mathbb{P}_\Theta(d\boldsymbol{\theta}) \le \|\mathbf{M} - \mathbf{M}'\|$.

ii) Consider $\Delta\beta_{bc}$ for two arbitrary distinct actions $b, c \in \mathcal{A}$ with $b \ne c$. Fix $\boldsymbol{\theta} \in \Delta\beta_{bc}$: action b is the optimal action for this type $\boldsymbol{\theta}$ in the state $\mathbf{M}$ and c is the optimal in the state $\mathbf{M}'$. Then,

$$0 \le \sum_{j\in\mathcal{A}\setminus\{b\}} Q_{jb}(\boldsymbol{\theta})\mu_j(\boldsymbol{\theta}) - \left(-Q'_{bc}(\boldsymbol{\theta})\mu'_b(\boldsymbol{\theta})\right) = \Delta v_b(\boldsymbol{\theta}) \le \sum_{j\in\mathcal{A}\setminus\{b\}} \bar{Q}\cdot 3 + \bar{Q}\cdot 3 = 3A\bar{Q}.$$

Similarly, we have $0 \ge \Delta v_c(\boldsymbol{\theta}) \ge -3A\bar{Q}$. For $i \ne b, c$,

$$\Delta v_i(\boldsymbol{\theta}) = \left(-Q_{ib}(\boldsymbol{\theta})\mu_i(\boldsymbol{\theta})\right) - \left(-Q'_{ib}(\boldsymbol{\theta})\mu'_i(\boldsymbol{\theta})\right).$$

Since $Q(\cdot) \in [0, \bar{Q}]$ and $\mu_\cdot(\cdot) \in [-3, 3]$, we have

$$|\Delta v_i(\boldsymbol{\theta})| \le |Q_{ib}(\boldsymbol{\theta})\mu_i(\boldsymbol{\theta})| + |Q'_{ib}(\boldsymbol{\theta})\mu'_i(\boldsymbol{\theta})| \le 6\bar{Q}.$$

Therefore,

$$\sum_{a\in\mathcal{A}} |\Delta v_i(\boldsymbol{\theta})| \le 2\cdot 3A\bar{Q} + (A-2)\cdot 6\bar{Q} = 12(A-1)\bar{Q}.$$

By Assumption 3, we have

$$\int_{\Delta\beta_{bc}} \sum_{a\in\mathcal{A}} |\Delta v_a(\boldsymbol{\theta})|\mathbb{P}_\Theta(d\boldsymbol{\theta}) \le 12(A-1)\bar{Q}\mathbb{P}_\Theta(\Delta\beta_{bc}) \le 12(A-1)\bar{Q}L_\beta|\bar{\mathbf{m}} - \bar{\mathbf{m}}'|$$

and thus

$$\begin{aligned}
&\int_{\Delta\beta} \sum_{a\in\mathcal{A}} |\Delta v_a(\boldsymbol{\theta})|\mathbb{P}_\Theta(d\boldsymbol{\theta}) = \sum_{b\in\mathcal{A}} \sum_{c\in\mathcal{A}\setminus\{b\}} |\Delta v_a(\boldsymbol{\theta})|\mathbb{P}_\Theta(d\boldsymbol{\theta}) \\
&\le 12A(A-1)^2\bar{Q}L_\beta|\bar{\mathbf{m}} - \bar{\mathbf{m}}'| \le 12A(A-1)^2\bar{Q}L_\beta\|\mathbf{M} - \mathbf{M}'\|. \qquad \text{(S.5)}
\end{aligned}$$

Again, the second inequality comes from $|\bar{\mathbf{m}} - \bar{\mathbf{m}}'| \le \|\mathbf{M} - \mathbf{M}'\|$.

As $\Theta = \cap\beta + \Delta\beta + N$ and $\mathbb{P}_\Theta(N) = 0$, we have

$$\|\mathbf{V}[\mathbf{M}] - \mathbf{V}[\mathbf{M}']\| = \int_{\cap\beta} \sum_{a\in\mathcal{A}} |\Delta v_a(\boldsymbol{\theta})|\mathbb{P}_\Theta(d\boldsymbol{\theta}) + \int_{\Delta\beta} \sum_{a\in\mathcal{A}} |\Delta v_a(\boldsymbol{\theta})|\mathbb{P}_\Theta(d\boldsymbol{\theta}).$$

(S.4) and (S.5) imply $\|\mathbf{V}[\mathbf{M}] - \mathbf{V}[\mathbf{M}']\| \le L_V\|\mathbf{M} - \mathbf{M}'\|$ with

$$L_V := 2(A-1)L_{\boldsymbol{\mu}}\{3L_Q\bar{L}_{\mathbf{F}} + (3L_Q + \bar{Q})\} + 12A(A-1)^2\bar{Q}L_\beta.$$

□

Consider an ASAG. Then, $\boldsymbol{\theta} \in \Delta\beta_{bc}$ is equivalent to

$$\begin{cases} F_b^0(\bar{\boldsymbol{\mu}}) + \theta_b & > F_j^0(\bar{\boldsymbol{\mu}}) + \theta_j \quad \text{for all } j \in \mathcal{A} \setminus \{b\} \\ F_c^0(\bar{\boldsymbol{\mu}}') + \theta_c & > F_j^0(\bar{\boldsymbol{\mu}}') + \theta_j \quad \text{for all } j \in \mathcal{A} \setminus \{c\} \end{cases}$$

This implies

$$F_c^0(\bar{\boldsymbol{\mu}}') - F_b^0(\bar{\boldsymbol{\mu}}') > \theta_b - \theta_c > F_c^0(\bar{\boldsymbol{\mu}}) - F_b^0(\bar{\boldsymbol{\mu}}).$$

Hence, if there exists $\bar{p}_\Theta \in \mathbb{R}$ such that $\mathbb{P}_\Theta(\{\boldsymbol{\theta} \in \Theta : c \leq \theta_b - \theta_a \leq d\}) \leq (d-c)\bar{p}_\Theta$, we have

$$\begin{aligned} \mathbb{P}_\Theta(\Delta\beta_{bc}) &\leq \bar{p}\left\{(F_c^0(\bar{\boldsymbol{\mu}}') - F_b^0(\bar{\boldsymbol{\mu}}')) - (F_c^0(\bar{\boldsymbol{\mu}}) - F_b^0(\bar{\boldsymbol{\mu}}))\right\} \\ &\leq 2\bar{p}\left\{|F_c^0(\bar{\boldsymbol{\mu}}') - F_c^0(\bar{\boldsymbol{\mu}})| + |F_b^0(\bar{\boldsymbol{\mu}}') - F_b^0(\bar{\boldsymbol{\mu}})|\right\} \\ &\leq 2\bar{p}L_F L_{\boldsymbol{\mu}}|\bar{\mathbf{m}} - \bar{\mathbf{m}}'|. \end{aligned} \tag{S.6}$$

Thus, Assumption 3 is satisfied.

Proof of part ii of Theorem 1

For part ii, we use Theorem 9 in Appendix A.3, namely, Zeidler (1986, Corollary 3.9).

Proof of parts ii. We leave only the boundedness of the dynamic; it comes from Assumption 2. Using the formula A.3 of the variational norm, we can obtain $\|\mathbf{V}^{\mathbf{F}}[\mathbf{M}]\| \leq 3A\bar{R}$ for all $\mathbf{M} \in \mathcal{M}$ since $v_i^{\mathbf{F}}[\tilde{M}](\boldsymbol{\theta}) \in [-3\bar{R}, 3\bar{R}]$ by Assumption 2 and $\mu(\cdot) \in [-3, 3]$.

Then, Theorem 9 implies the unique existence of a solution path of the dynamic on $\mathcal{M}$. Notice that $\mathcal{X}$ is forward invariant under $\mathbf{V}$. Therefore, if the initial state $\mathbf{X}_0$ lies in $\mathcal{X} \subset \mathcal{M}$, then the unique solution that passes $\mathbf{X}_0$ at time 0 should remain in $\mathcal{X}$. □