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A unifying and intuitive approach to linking static and dynamic stability

Dai Zusai
Department of Economics
Temple University

Department of Economics
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A unifying and intuitive approach to linking static and dynamic stability*

Dai ZUSAI†

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†Department of Economics, Temple University, 1301 Cecil B. Moore Ave., RA 873 (004-04), Philadelphia, PA 19122, U.S.A. Tel:+1-215-204-8880. E-mail: `zusai@temple.edu`.

Abstract

This paper presents a universal and economically intuitive approach to linking static and dynamic stability. In economics, static stability is traditionally defined by negative relationship between endogenous and exogenous variables in a model: for a population game, this is characterized by negative semidefiniteness of the Jacobian matrix of the payoff function. We consider *economically reasonable dynamics*, in which we can justify agents' choices of new strategies as optimal choices possibly by introducing additional costs and constraints. This class of dynamics covers major payoff-based (non-imitative) evolutionary dynamics. The key is expected net gains (payoff improvements) from strategy revisions after paying switching costs. Static stability implies that the aggregate net gain diminishes over time under economic reasonable dynamics and thus can be used as a Lyapunov function. While our analysis here is confined to myopic evolutionary dynamics in population games, our approach is applicable to more complex situations. (149 words)
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JEL classification: C73, C62, C61

Greed is right. Greed works. Greed clarifies, cuts through, and captures the essence of the evolutionary spirit.—Gordon Gekko, *Wall Street* (1987)

1 Introduction

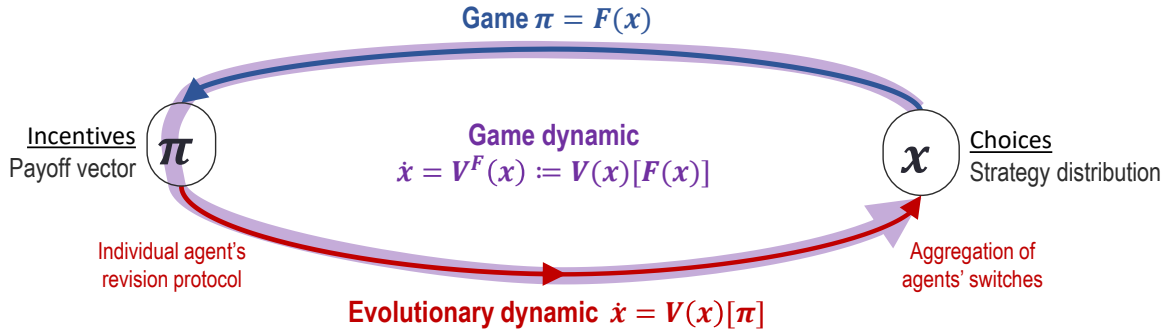
Equilibrium stability has been a classic topic in economic theory.[1] Stability assures that an equilibrium is robust to small perturbation or misspecification and also we can obtain approximation of equilibria from iterative simulation of the model. An economic model formulates relationship between choices and incentives ultimately with a system of functions that typically take either one side as exogenous variables and the other as endogenous variables: a strategy profile/distribution determines payoffs of strategies in a game theoretic model through a payoff function, and prices determine excess demand in a competitive market model through an excess demand function. To pin down both exogenous and endogenous variables, an equilibrium in a model is defined as a solution of a system of equations such as equality between payoffs of optimal strategies in a game[2] and zero excess demand in a market. Economists argue whether an economic system reaches this mathematical solution from a natural economic process and how robust it is to deviation from equilibrium.

Conventional economists develop the notion of *static stability* in various situations based on economic intuition; basically, static stability refers to negative response of endogenous variables to deviations in exogenous variables from equilibrium.[3] Economists *expect* it to adjust back the latter variables through natural economic driving forces—not modeled explicitly—and bring the economy back to an equilibrium. Math-oriented theorists explicitly formulate such adjustment processes and investigate dynamic stability based on mathematical theory of dynamic

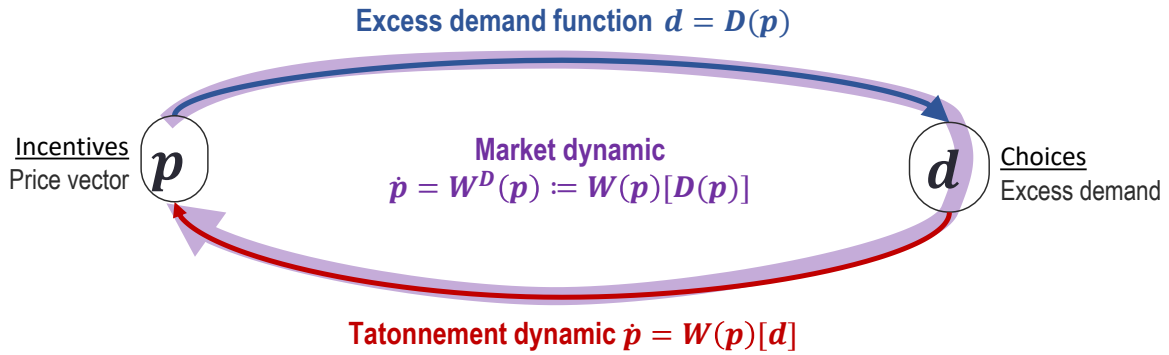
[1] Weintraub (1991) gives an excellent bibliographical study on the development of the stability notion in economics, especially in the context of general equilibrium models.

[2] Here we imagine a completely mixed strategy Nash equilibrium. A boundary equilibrium is identified not only by incentives but also by feasibility constraints (non-negativity of probabilities) and thus needs additional consideration. It will be discussed in Section 5.2.

[3] In our context of large population games in the normal form, the notion of evolutionary stable states (ESS) is the dominant concept of local static stability; a global version is contractive games (Hofbauer and Sandholm, 2009). In the classic competitive market model, Hicksian stability (Hicks, 1939) is the seminal concept. In game theoretical models, economists develop various static stability concepts à la ESS, tailor-made for each specific model; e.g. Fujiwara-Greve and Okuno-Fujiwara (2009); Dekel et al. (2007); Alger and Weibull (2013); Demuynck et al. (2019). Our approach helps them to prove dynamic stability of these ad hoc static stability based on economic intuition, free from specification of dynamics; see Section 7.



(a) Game theoretical models.



(b) Competitive market model.

Figure 1: Dynamics in economic models. Static stability is essentially negative response in the functional relationship that represents the model, such as **F** or **D** above. Dynamic stability is argued with mathematical rigor by explicitly formulating the adjustment dynamic such as **V** or **W** and completing the loop.

systems; then, dynamic stability is rigorously examined by checking convergence to equilibrium. Despite innocent hope to link static and dynamic stability, those theorists have found counterexamples to show dynamic *in*stability of statically stable equilibria; e.g. Friedman (1991) on evolutionary stable states and Samuelson (1941) on Hicksian stability of Walrasian equilibrium. On the other hand, conventional economists may be skeptical about economic insights behind purely mathematical investigation of dynamic stability and generality of results beyond mathematical specification, as classically expressed by Hicks (1939).[4]

This paper tries to bring back them in harmony by presenting an *economically natural* approach to verifying dynamic stability from static stability with *mathemat-*

[4]Hicks (1939, Additional note C) criticized Samuelson's work on dynamic stability as it missed shedding light on "the behavior of people and the motives of their conduct," which Hicks claimed to reflect on his notion of static stability.

ical rigor in the context of deterministic evolutionary dynamics in large population games. We concretely present our approach by proving global stability of the Nash equilibrium set in contractive games and local stability of an evolutionary stable state (ESS) under a general class of payoff-based (non-imitative) evolutionary dynamics. While these stability results have been proven for *each* of major evolutionary dynamics separately,[5] we find a single *universally applicable* and also *economically intuitive* idea of the proof that encompasses various evolutionary dynamics. The single universal proof guarantees that we can extend these stability results to heterogeneous social dynamics in which different agents follow different decision rules and have different payoff functions. Besides, construction of the proof based on economic intuition promises that we can expect dynamic stability without identifying the dynamic quantitatively as long as agents' decision rules meet a few economic and informational assumptions. While we restrict our attention to evolution in the strategic form of a game, our approach is applicable to more complex situations with multilayered or non-myopic decisions.[6]

To set a goal, we first pin down the notion of *static stability*. In our context, it means that, when the strategy distribution deviates from an equilibrium, payoffs are adjusted in the game to make favor of actions with smaller shares than in equilibrium. Mathematically it is characterized by negative semidefiniteness of the Jacobian matrix of a payoff function. This has indeed appeared in the definition of contractive games and interior (and regular) evolutionary stable states. It is analogous to (Hicksian) static stability in market models, which essentially means negative semidefiniteness of the Jacobian matrix of an excess demand function.

Economic intuition leads us to expecting naturally that static stability should assure a return to equilibrium, i.e., dynamic stability, as long as the dynamic is "economically reasonable." However, there are counterexamples (Friedman, 1991; Hofbauer and Sandholm, 2009; Milchtaich, 2019) in which incentive compatible dynamics, which is typically defined by each author as some sorts of payoff monotonicity, fail to converge to an interior ESS.[7] Thus, we need to redefine economic

[5]See Hofbauer and Sandholm (2009) and Fox and Shamma (2013) for the recent study on convergence results in contractive games and Sandholm (2010a) for local stability of an ESS.

[6]Sawa and Zusai (2019) is indeed an application of our approach to a multilayered decision in a multitasking environment. The dynamic stability proven in the current paper is also extended directly to forward-looking decision with adaptive expectation as in Fox and Shamma (2013). We further explore this possibility in Section 7.

[7]On equilibrium stability of contractive games, Hofbauer and Sandholm (2009) argue that "con-

reasonability of evolutionary dynamics.

What is "economic reasoning"? According to any principle-level microeconomics textbooks, optimization is the very first principle of economic reasoning.[8] It is also prevalent in both theoretical and empirical research in economics, as argued in Section 7. On the other hand, evolutionary game theory admits deviation from exact optimization, while imposing more or less of consistency with incentives. Agents may choose suboptimal strategies, or may have inertia to prevent them from utilizing a given opportunity of a switch to a better strategy when the payoff difference is small. How can we reconcile them? Again, any principle textbook admits such distortion and then justifies it by arguing implicit/hidden costs and constraints due to behavioral/psychological factors, informational frictions, etc. We can actually justify apparently suboptimal choices and inertia by introducing restriction to available strategies and costs of switching strategies.

All these things accounted, we say an evolutionary dynamic is *economically reasonable* if the agents' switching decision can be justified from optimization by introducing those additional costs and constraints. The class of economically reasonable dynamics includes not only the best response dynamic (BRD) and its variant (tempered and perturbed) but also other major dynamics, such as pairwise comparison dynamics and excess payoff dynamics, as well as interesting new dynamics.[9]

We prove dynamic stability under this class of dynamics mathematically by utilizing the theory of a Lyapunov function. A *Lyapunov function* is a mapping from a position of the current state to a scalar value; the value should decrease monotonically over time until the dynamic reaches a limit state. In kinetics, it is the height of an object; an object must fall down by gravity until it reaches the ground.

In economics, what can be a Lyapunov function? What can be the gravity in economic dynamics? Again, economic intuition would guide us to looking into *greed*, or the tendency to switch to a better option as the essential driving force behind dynamism of the economy. Whenever there is a room to gain from arbitrage, agents should exploit it by changing their choices. Static stability implies that these changes

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vergence results for stable [contractive] games are not as general as those for potential games" and found "[payoff] monotonicity of the dynamics" is not sufficient; as a needed additional condition on dynamics, they assumed a sort of integrability on revision protocols.

[8] To name a few, see Acemoglu et al. (2018, Sec. 1.3) and Hubbard and O'Brien (2019, Sec. 1.1).

[9] Yet, we exclude imitative dynamics, about which it is already known that static stability does not guarantee dynamic stability; we argue in Section 6.3 that those dynamics indeed behave against economic intuition and thus cannot be explained simply by incentives.

in their choices never refuel the remaining available gains. If there is no room for a gain, it means an equilibrium in the economic sense. Thus, we can use aggregate remaining gains from revisions as a candidate for a Lyapunov function.

However, if we simply count payoff improvements by revisions, it does not work—it does not monotonically diminish, since evolutionary agents may switch to suboptimal choices and may leave some possible gains unexploited. However, our framework justifies such distortions from exact optimization economically by revealing additional costs and constraints. Taking these into account, we define the notion of an ex-ante *net gain* as the maximal payoff improvements subject to the constraints on available strategies minus the additional costs. The aggregate ex-ante net gain works as a Lyapunov function generally even under heterogeneity.

In this paper, we start from a simplified, basic model in order to give a clear presentation of our approach. We consider a single-population game and assume that an agent chooses a pure strategy (an action), as typical in deterministic evolutionary dynamics in a large population. This basic model covers the BRD and pairwise comparison dynamics. Excess payoff dynamics and perturbed BRDs are covered by modifying the model to fit them better to our concept of economic reasonability.

Our analysis goes step by step. We present the basic set-up in Section 2 and summarize the notions of static and dynamic stability in Section 3. In these sections, separation of an evolutionary dynamic and a game is emphasized as a crucial perspective for our analysis. In Section 4, we reach the main conclusion of the paper, i.e., the link between static and dynamic stability through the net gain. Section 5 collects extensions of this result beyond the basic set-up. In Section 6, we consider other would-be approaches, e.g., using the gross gain for a Lyapunov function, imposing a kind of payoff monotonicity instead of our notion of economic reasonability, and considering imitative dynamics, and see why they do *not* work. In Section 7, we explore implications in different contexts from past history to future applications, while giving a quick summary of our approach. The proofs of main results in Section 4 are given in Appendix. Details of side issues are given in Supplement.

2 Basic set-up

We first set up a game and an evolutionary dynamic separately. A (population) game $\mathbf{F} : \mathbf{x} \mapsto \boldsymbol{\pi}$ is ultimately a functional relationship that determines payoff vec-

tor $\boldsymbol{\pi}$ from strategy distribution $\mathbf{x}$; it represents an economic environment. An evolutionary dynamic $\mathbf{V} : (\mathbf{x}, \boldsymbol{\pi}) \mapsto \dot{\mathbf{x}}$ is mathematically a dynamic system that determines the transition of strategy distribution $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})[\boldsymbol{\pi}]$ given the current state of $\mathbf{x}$ and $\boldsymbol{\pi}$; this describes how agents adjust their choices to the payoffs. Combining $\mathbf{F}$ and $\mathbf{V}$, we can close a loop and obtain an autonomous dynamic system $\mathbf{V}^{\mathbf{F}} : \mathbf{x} \mapsto \dot{\mathbf{x}}$ as $\mathbf{V}^{\mathbf{F}}(\mathbf{x}) \coloneqq \mathbf{V}(\mathbf{x})[\mathbf{F}(\mathbf{x})]$ that determines $\dot{\mathbf{x}}$ from $\mathbf{x}$; we call it a game dynamic.[10] Static stability is a property of game $\mathbf{F}$, while dynamic stability is of autonomous game dynamic $\mathbf{V}^{\mathbf{F}}$. Our goal is to find a general economic principle behind $\mathbf{V}$ to link static stability of $\mathbf{F}$ and dynamic stability of $\mathbf{V}^{\mathbf{F}}$. Therefore, separation of $\mathbf{V}$ and $\mathbf{F}$ is crucial for our study.

Remark. Sandholm (2010b) is a manifesto of constructing evolutionary dynamic $\mathbf{V}$ from a revision protocol and regarding $\mathbf{V}$ as an operation to generate an autonomous dynamic system $\mathbf{V}^{\mathbf{F}}$ from a game $\mathbf{F}$. In line with this, Shamma (2016) proposes a separatist view as above. In the study of equilibrium stability in contractive games, Fox and Shamma (2013) push this approach further by decomposing a mathematical property of $\mathbf{V}^{\mathbf{F}}$ that contributes to equilibrium stability into properties of $\mathbf{V}$ and of $\mathbf{F}$ (see Section 3.3).[11] The analysis in this paper stems from such separatists' contemporary view of evolutionary dynamics.

2.1 Population game F

We consider a finite-action population game played in the society of continuously many homogeneous agents.[12] The society consists of a unit mass of agents. In this basic set-up, they are homogeneous in the sense that they have the same action set and the same payoff function.

More specifically, each agent chooses an action a from set $\mathcal{A} \coloneqq \{1, \dots, A\}$. Denote by $x_a \in [0, 1]$ the mass of action-a players in the society. The **social state** is represented by a column vector $\mathbf{x} \coloneqq (x_1, \dots, x_A)$ in $\mathcal{X} \coloneqq \Delta^{\mathcal{A}}$.[13] The tangent space

[10] When we argue evolutionary dynamic $\mathbf{V}$, we regard $\boldsymbol{\pi}$ as a parameter of the dynamic and thus put it in square brackets, separated from $\mathbf{x}$.

[11] See also Mabrok and Shamma (2016). Unlike our study here, Fox and Shamma (2013) are interested in general mathematical properties (δ-anti-passivity) of $\mathbf{F}$ to guarantee the equilibrium stability under each of major evolutionary dynamics.

[12] For further details, see Sandholm (2010b, Ch.2).

[13] The vector in a bold font like $\mathbf{v}$ is a column vector, while the one with an arrow over the letter like $\vec{v}$ is a row vector. We omit the transpose when we write a column vector on the text. $\mathbf{1}$ is a column vector $(1, 1, \dots, 1)$. Note that $\mathbf{1} \cdot \mathbf{z} = \sum_{i=1}^{n} z_i$ for an arbitrary column vector $\mathbf{z} = (z_i)_{i=1}^{n} \in \mathbb{R}^n$.

of $\mathcal{X}$ is $T\mathcal{X} \coloneqq \mathbb{R}^A_0$. The payoff of each action is a function of the social state; in social state $\mathbf{x} \in \mathcal{X}$, $F_a(\mathbf{x}) \in \mathbb{R}$ is the payoff from action $a \in \mathcal{A}$. Define payoff function $\mathbf{F} : \mathcal{X} \to \mathbb{R}^A$ by $\mathbf{F}(\mathbf{x}) \coloneqq (F_a(\mathbf{x}))_{a\in\mathcal{A}}$ for each $\mathbf{x} \in \mathcal{X}$. Throughout this paper, we assume that $\mathbf{F}$ is continuously differentiable. Let $F_*(\mathbf{x}) \coloneqq \max_{a\in\mathcal{A}} F_a(\mathbf{x})$. A population game is characterized by $(\mathcal{A}, \mathbf{F})$, or shortly by $\mathbf{F}$.

As usual, a Nash equilibrium is a state where (almost) every agent takes an optimal action. Formally, social state $\mathbf{x}^* \in \mathcal{X}$ is a **Nash equilibrium** if $F_a(\mathbf{x}^*) = F_*(\mathbf{x}^*)$ whenever $x^*_a > 0$, or equivalently

$$(\mathbf{x} - \mathbf{x}^*) \cdot \mathbf{F}(\mathbf{x}^*) \leq 0 \qquad \text{for all } \mathbf{x} \in \mathcal{X}. \tag{1}$$

Denote by $\mathrm{NE}(\mathbf{F})$ the set of Nash equilibria in population game $\mathbf{F}$.

The simplest example of a population game is random matching in a symmetric two-player normal-form game with a payoff matrix $\mathbf{\Pi}$; the population game is given by $\mathbf{F}(\mathbf{x}) = \mathbf{\Pi}\mathbf{x}$. $\mathrm{NE}(\mathbf{F})$ coincides with the set of *symmetric* Nash equilibria of $\mathbf{\Pi}$.

2.2 Evolutionary dynamic V

General formulation of economically reasonable dynamics

Here we construct an evolutionary dynamic to formulate transition of the social state over continuous time horizon. In the dynamic, each agent occasionally receives an opportunity to revise the action, following the Poisson process with arrival rate 1. There is a wide variety of evolutionary dynamics stemmed from a variety of revision protocols (decision rules). Here we reduce them to a unifying framework based on optimization, while keeping flexibility of the framework to recover the variety.[14]

Let's set up an optimization-based framework of revision protocols. Upon receipt of a revision opportunity, a revising agent compares payoffs from the current action and from other *available* actions. Given current action a, all other actions may not be always available: available action set $\mathcal{A}'_a$ is a subset of $\mathcal{A} \setminus \{a\}$. Besides, if the revising agent chooses to switch an action, the agent needs to pay *switching cost* q.

For a finite set $\mathcal{A} = \{1, \ldots, A\}$, we define $\Delta^{\mathcal{A}}$ as $\Delta^{\mathcal{A}} \coloneqq \{\mathbf{x} = (x_a)_{a\in\mathcal{A}} \in [0,1]^A \mid \mathbf{1} \cdot \mathbf{x} = 1\}$, i.e., the set of all probability distributions on $\mathcal{A}$, and $\mathbb{R}^A_0 \coloneqq \{\mathbf{z} \in \mathbb{R}^A \mid \mathbf{1} \cdot \mathbf{z} = 0\}$.

[14] On stochastic adaptive learning in finite normal-form games, Funai (2018) proposes a framework to unify various learning models such as stochastic fictitious play, experience-weighted attraction learning, etc.; then, he investigates how convergence to a quantal response equilibrium depends on these learning parameters in several classes of games.

The switching cost does not depend on which of available actions to switch to. Observing payoff vector $\boldsymbol{\pi} \in \mathbb{R}^A$, the agent chooses the action that yields the greatest net payoff:

$$\max \left\{ \pi_a, \max_{b \in \mathcal{A}'_a} \pi_b - q \right\} \tag{2}$$

Let $\pi_*(\mathcal{A}'_a) := \max_{b \in \mathcal{A}'_a} \pi_b$ be the greatest (gross) payoff among available actions $\mathcal{A}'_a$, and $b_*[\boldsymbol{\pi}](\mathcal{A}'_a) := \operatorname{argmax}_{b \in \mathcal{A}'_a} \pi_b$ be the set of the optimal actions among them, while $\pi_* := \max_{b \in \mathcal{A}} \pi_b$ denotes the greatest payoff among all the actions $\mathcal{A}$ and $b_*[\boldsymbol{\pi}] := \max_{b \in \mathcal{A}} \pi_b$ the set of the optimal actions. Note that $\mathbf{x} \in \mathrm{NE}(\mathbf{F})$ is equivalent to $\mathbf{x} \in \Delta^{\mathcal{A}}(b_*[\mathbf{F}(\mathbf{x})])$.[15]

We allow both available action set $\mathcal{A}'_a$ and switching cost q to be randomly assigned after the receipt of the revision opportunity. Available action set $\mathcal{A}'_a \subset \mathcal{A} \setminus \{a\}$ is drawn from probability distribution $\mathbb{P}_{\mathcal{A}a}$ over $2^{\mathcal{A} \setminus \{a\}}$, the power set of $\mathcal{A} \setminus \{a\}$. Switching cost $q \in \mathbb{R}_+$ is drawn from probability distribution $\mathbb{P}_Q$ over $\mathbb{R}_+$; let $Q_-(q) = \mathbb{P}_Q((-\infty, q))$ and $Q(q) = \mathbb{P}_Q((-\infty, q])$. We assume $\mathbb{P}_Q$ has no singular continuous part.[16] Let $\mathcal{Q}(q) := [Q_-(q), Q(q)] \subset [0, 1]$. Note that, if $\mathbb{P}_Q$ is continuous at q, i.e., $Q_-(q) = Q(q)$, then $\mathcal{Q}(q)$ reduces to a singleton $\{Q(q)\}$. While we distinguish $\mathcal{Q}$ from Q for mathematical completeness in case $\mathbb{P}_Q$ is not continuous, readers do not have to be bothered when reading the main text; when it matters anyhow, it is cautioned explicitly. We also assume that $\mathcal{A}'_a$ and q are independent of each other, the agent's current action, and the social state, and that $\mathbb{P}_{\mathcal{A}a}$ and $\mathbb{P}_Q$ are also invariant over time.

The revising agent chooses to switch to either one action in $b_*[\boldsymbol{\pi}](\mathcal{A}'_a)$ if $\pi_*(\mathcal{A}'_a) - q$ is greater than π_a, and to maintain the current action a as the status quo if it is smaller; these choices are indifferent if it is equal to π_a. Hence, the probability of choosing a switch is in the range $\mathcal{Q}(\pi_*(\mathcal{A}'_a) - \pi_a)$; we do not impose any tie-breaking rule. We can interpret $\pi_*(\mathcal{A}'_a) - q - \pi_a$ as the **net gain** from the switch; in contrast, we call $\pi_*(\mathcal{A}'_a) - \pi_a$ the **gross gain**.

Given that a revising agent was taking action a until this revision opportunity, the probability distribution of possible changes in her action upon the revision op-

[15] Consider a $|\mathscr{U}|$-dimensional real space, each of whose coordinate is labeled with either one element of $\mathscr{U} = \{1, \ldots, |\mathscr{U}|\}$. For a set $S \subset \mathscr{U}$, we denote by $\Delta^{|\mathscr{U}|}(S)$ the set of probability vectors whose support is contained in S: i.e., $\Delta^{|\mathscr{U}|}(S) := \left\{ \mathbf{x} \in \mathbb{R}_+^{|\mathscr{U}|} \mid \sum_{k \in S} x_k = 1 \text{ and } x_l = 0 \text{ for any } l \in \mathscr{U} \setminus S \right\}$.

[16] That is, $\mathbb{P}_Q$ is a convex combination of an absolute continuous measure and a discrete measure.

portunity takes a form of a probability vector such that[17]

$$\sum_{\mathcal{A}'_a \subset \mathcal{A} \setminus \{a\}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) Q_a(\mathcal{A}'_a) \left\{ \mathbf{y}_a(\mathcal{A}'_a) - \mathbf{e}_a \right\} \in T\mathcal{X}.$$

Here, $Q_a(\mathcal{A}'_a) \in \mathcal{Q}(\pi_*(\mathcal{A}'_a) - \pi_a)$ is a probability of switching to a new strategy, conditional on the receipt of a revision opportunity and the draw of $\mathcal{A}'_a$; we call it a *conditional switching rate*. Besides, $\mathbf{y}_a(\mathcal{A}'_a) \in \Delta^{\mathcal{A}}(b_*[\boldsymbol{\pi}](\mathcal{A}'_a))$ is a probability vector that assigns a positive probability only to optimal available actions, i.e., actions in $b_*[\boldsymbol{\pi}](\mathcal{A}'_a)$. If $b_*[\boldsymbol{\pi}](\mathcal{A}'_a)$ is not a singleton, $\mathbf{y}_a(\mathcal{A}'_a)$ cannot be determined uniquely. Because of these multiplicities, we formulate the dynamic as a set-valued differential equation (differential inclusion). Denote the set of vectors in the above form by[18]

$$\mathbf{V}_a[\boldsymbol{\pi}] \coloneqq \sum_{\mathcal{A}'_a \subset \mathcal{A} \setminus \{a\}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) \mathcal{Q}(\pi_*(\mathcal{A}'_a) - \pi_a) \left\{ \Delta^{\mathcal{A}}(b_*[\boldsymbol{\pi}](\mathcal{A}'_a)) - \mathbf{e}_a \right\}.$$

Aggregating $\mathbf{V}_a[\boldsymbol{\pi}]$ over the whole population, the transition of the social state is represented by a vector $\dot{\mathbf{x}} \in T\mathcal{X}$ in the set $\mathbf{V}(\mathbf{x})[\boldsymbol{\pi}] = \sum_{a \in \mathcal{A}} x_a \mathbf{V}_a[\boldsymbol{\pi}]$: i.e,

$$\dot{\mathbf{x}} \in \mathbf{V}(\mathbf{x})[\boldsymbol{\pi}] \iff \left[\forall\, a \in \mathcal{A}\ \exists\, \mathbf{z}_a \in \mathbf{V}_a[\boldsymbol{\pi}] \text{ s.t. } \dot{\mathbf{x}} = \sum_{a \in \mathcal{A}} x_a \mathbf{z}_a \right]. \tag{3}$$

This $\mathbf{V} : \mathcal{X} \times \mathbb{R}^A \to T\mathcal{X}$ is an **evolutionary dynamics**.

Assumptions

Our formulation itself imposes some restrictions. One significant restriction is that the distribution of available action sets $\mathbb{P}_{\mathcal{A}\cdot}$ does not depend on the social state $\mathbf{x}$. This excludes imitative dynamics such as the replicator dynamic. Actually, it is known that dynamic stability under imitative dynamics cannot be obtained from static stability; see Section 6.3. We make this restriction explicit as follows.

Assumption A0. $\mathbb{P}_{\mathcal{A}\cdot}$ is independent of $\mathbf{x}$.

About the distribution of switching costs, we first assume the following.

[17]Here $\mathbf{e}_a = (e_{ab})_{b \in \mathcal{A}} \in \Delta^{\mathcal{A}}$ is a unit vector such that $e_{aa} = 1$ and $e_{ab} = 0$ for all $b \neq a$. This represents a population state where every agent takes action a.

[18]In a vector space $\mathcal{Z}$, with a set $S \subset \mathcal{Z}$, an element $\mathbf{c} \in \mathcal{Z}$ and a scalar $k \in \mathbb{R}$, we define set $k(S + \mathbf{c}) \coloneqq \{k(\mathbf{z} + \mathbf{c}) \in \mathcal{Z} \mid \mathbf{z} \in S\}$; with a scalar set $K \subset \mathbb{R}$, set KS is defined as $KS \coloneqq \{k\mathbf{z} \in \mathcal{Z} \mid k \in K, \mathbf{z} \in S\}$. For sets $S_1, S_2 \subset \mathcal{Z}$, we define set $S_1 + S_2 \coloneqq \{\mathbf{z}_1 + \mathbf{z}_2 \in \mathcal{Z} \mid \mathbf{z}_1 \in S_1 \text{ and } \mathbf{z}_2 \in S_2\}$.

Assumption Q1. i) $Q(q) = 0$ if $q < 0$ and ii) $Q(q) > 0$ if $q > 0$.[19]

The assumption guarantees that switching cost is nonnegative almost surely and an agent chooses to switch her action with some positive probability whenever there is an available action strictly better than the current action. Together with Assumption A1-i) below, Assumption Q1 guarantees stationarity of a Nash equilibrium.

The following assumptions are essential, despite its notational complication, to make myopic evolutionary dynamics economically natural and to obtain a natural relationship between payoffs and the notion of gains.

Assumption A1-i). For any $a \in \mathcal{A}$ and any $b \in \mathcal{A} \setminus \{a\}$, $\mathbb{P}_{\mathcal{A}a}(\{\mathcal{A}'_a \subset \mathcal{A} \setminus \{a\} \mid b \in \mathcal{A}'_a\}) > 0$.

Assumption A1-ii). For any $a, b \in \mathcal{A}$ and any $\mathcal{A}_{ab} \subset \mathcal{A} \setminus \{a, b\}$, $\mathbb{P}_{\mathcal{A}a}(\{\mathcal{A}'_a \subset \mathcal{A} \setminus \{a\} \mid \mathcal{A}'_a \cap \mathcal{A}_{ab} \neq \emptyset\}) = \mathbb{P}_{\mathcal{A}b}(\{\mathcal{A}'_b \subset \mathcal{A} \setminus \{b\} \mid \mathcal{A}'_b \cap \mathcal{A}_{ab} \neq \emptyset\})$.

Assumption A1-i) requires any action to be available with some positive probability. Assumption A1-ii) imposes invariance of availability of actions other than the current action. Specifically, we compare the distribution of available actions $\mathbb{P}_{\mathcal{A}a}$ when the status-quo (current) action is a and the one $\mathbb{P}_{\mathcal{A}b}$ when it is b. Fix an arbitrary subset $\mathcal{A}'_{ab}$ of actions other than a and b. The assumption requires $\mathbb{P}_{\mathcal{A}a}$ and $\mathbb{P}_{\mathcal{A}b}$ to assign the same probability on the event that at least one of the actions in $\mathcal{A}'_{ab}$ becomes available.[20]

Assumption A1-ii) implies that, the better action an agent chooses, the smaller room of the payoff improvement remains for the next revision opportunity; this is one of important properties of the gain, to be stated in Theorem 2. If this assumption is not satisfied, the current action affects mobility to a further better action; then, economic intuition would suggest a revising agent to consider not only the payoff directly earned from a new action but also opportunity costs of changing mobility in the next revision opportunity. While it is interesting to consider such a dynamic, here we focus on myopic evolutionary dynamics where agents respond simply to the current payoffs and we are trying to find *economically natural* driving force behind them; to accomplish our goal, we exclude a situation where myopic decision is not well suited with economic intuition.

[19] This assumption is equivalent to i) $Q_-(q) = 0$ if $q \leq 0$ and ii) $Q_-(q) > 0$ if $q > 0$.

[20] A change of the invariance condition in either of these two aspects may break the monotone relationship between the gain ordering and the payoff ordering; see discussion and counterexamples in Supplement S3.

We say an evolutionary dynamic is **economically reasonable** if it is constructed as (3) from an optimization-based revision protocol (2) with $\mathbb{P}_Q$ satisfying Assumption Q1 and $\mathbb{P}_{A\cdot}$ satisfying Assumptions A0, A1-i) and A1-ii)). A variety of evolutionary dynamics is found to be economically reasonable by introducing random costs of switching and constraints on available actions to explain inertia against small improvement of payoffs and switches to better but not the best actions. This basic framework captures not only the standard best response dynamics (BRD; Gilboa and Matsui (1991); Hofbauer (1995)) but also tempered BRDs (Zusai, 2018) and pairwise comparison dynamics (Smith, 1984); see Figure 2. It also opens new evolutionary dynamics, such as a version of tempered BRDs where switching rates depend on the ordinal payoff ranking; see Supplement S1.1. Our *approach* covers excess payoff dynamics and smooth BRDs with a little modification of the framework, especially in the definition of a status-quo strategy for an agent, as discussed in Section 5.1.

2.3 Game dynamic $\mathbf{V}^{\mathbf{F}}$

Now we have set up both game $\mathbf{F} : \mathbf{x} \mapsto \boldsymbol{\pi}$ and evolutionary dynamic $\mathbf{V} : (\mathbf{x}, \boldsymbol{\pi}) \mapsto \dot{\mathbf{x}}$. Combining them, we obtain an autonomous dynamic system $\mathbf{V}^{\mathbf{F}} : \mathbf{x} \mapsto \dot{\mathbf{x}}$ to completely describe the transition of social state $\mathbf{x}$ as

$$\dot{\mathbf{x}} \in \mathbf{V}(\mathbf{x})[\mathbf{F}(\mathbf{x})] =: \mathbf{V}^{\mathbf{F}}(\mathbf{x}).$$

We call $\mathbf{V}^{\mathbf{F}}$ the **game dynamic**. Typically, evolutionary game theorists investigate mathematical properties of $\mathbf{V}^{\mathbf{F}}$ to study dynamics.

Our construction of an optimization-based revision protocol implies that a Nash equilibrium is a stationary state. Assumptions Q1 and A1-i) guarantee that a revising agent can always switch to an optimal action with some positive probability; so, a stationary state must be also a Nash equilibrium. As a result, we have equivalence between stationarity and Nash equilibrium, i.e., $\mathbf{0} \in \mathbf{V}^{\mathbf{F}}(\mathbf{x}) \Leftrightarrow \mathbf{x} \in \mathrm{NE}(\mathbf{F})$.[21]

[21] The formal proof for this can be easily obtained in a similar way to (Zusai, 2018, Theorem 1).

3 Static and dynamic stability

3.1 Static stability of F

As we argued in the introduction, an economic model is ultimately represented by a functional relationship between exogenous variables and endogenous variables. We have formulated a game by a payoff function $\mathbf{F} : \mathcal{X} \to \mathbb{R}^A$ that determines payoff vector $\boldsymbol{\pi} = \mathbf{F}(\mathbf{x}) \in \mathbb{R}^A$ from action distribution $\mathbf{x} \in \mathcal{X}$. Then, in equilibrium analysis, we pin down Nash equilibrium $\mathbf{x}^*$ by imposing (1).

Without explicitly considering the feedback adjustment of $\mathbf{x}$ in response to $\boldsymbol{\pi}$, conventional economists would believe that the economy should return to an equilibrium as long as there is negative correlation between deviations in action distribution $\Delta\mathbf{x} \in T\mathcal{X}$ and resulting changes in payoff vector $\Delta\boldsymbol{\pi}$. With a linear approximation of the latter by $\Delta\boldsymbol{\pi} \approx D\mathbf{F}(\mathbf{x})\Delta\mathbf{x}$, this is expressed as

$$\Delta\mathbf{x} \cdot D\mathbf{F}(\mathbf{x})\Delta\mathbf{x} \leq 0 \qquad \text{for any } \Delta\mathbf{x} \in T\mathcal{X} = \mathbb{R}^A_0. \tag{4}$$

The condition states that, on average, actions that have excessively greater shares than in the equilibrium should not yield greater payoffs than in the equilibrium, and actions with smaller shares should yield greater profits.[22] Economists expect incentive-driven economic agents to switch from the former to the latter and thus the society to return to the equilibrium.

This is, of course, nothing but negative semidefiniteness of $D\mathbf{F}$. Note that this imposes a condition only on $\mathbf{F}$, but not on $\mathbf{V}$. So we can regard it as *static* stability.[23]

Definition 1 (Static stability)**.** Consider a population game $\mathbf{F} : \mathcal{X} \to \mathbb{R}^A$. If $D\mathbf{F}(\mathbf{x})$ satisfies the negative semideniteness condition (4) at any $\mathbf{x} \in \mathcal{X}$, we say that the game satisfies **static stability** globally. If there is a neighborhood of an isolated Nash equilibrium $\mathbf{x}^* \in \mathcal{X}$ such that (4) is satisfied at any $\mathbf{x}$ in the neighborhood, then we say that the game satisfies static stability locally around $\mathbf{x}^*$.[24]

[22] Hofbauer and Sandholm (2009) call it *self-defeating externality*.

[23] Milchtaich (2019) finds such an idea of incentives of returning to an equilibrium behind our conventional thought of static stability, in his attempt to exactly *define* a static stability concept that is universally applicable to different settings (finite player games, population games, etc.). However, there are a few differences in the details between Milchtaich's static stability and what we extract here as "static stability" from contractive games and ESS. Here we look at the correlation between $\Delta\mathbf{x}$ and $\Delta\boldsymbol{\pi} = D\mathbf{F}(\mathbf{x}) \cdot \Delta\mathbf{x}$ at each state $\mathbf{x}$, while he imagines a path where agents linearly move from the current state $\mathbf{x}$ to an equilibrium and calculates the total payoff changes that occur on the path.

[24] We say a Nash equilibrium $\mathbf{x}^*$ is *isolated* if there is no other Nash equilibrium in a neighborhood

We do *not* assert the above static stability is a newly invented notion. Rather, even in game theory, it only summarizes what we have already known as a property of game **F** for stability. Global static stability of **F** is indeed equivalent to **F** being a contractive game (Hofbauer and Sandholm, 2009). Static stability is locally satisfied at an interior ESS; a boundary (regular) ESS satisfies a weaker version of static stability, as we argue in Section 5.2. See Supplement S2.1. Recall that a classical market model is ultimately represented by an excess demand function $\mathbf{d} = \mathbf{D}(\mathbf{p})$ that maps price vector $\mathbf{p}$ to excess demands $\mathbf{d}$; Hick's static stability condition is essentially the negative semidefiniteness of Jacobian $D\mathbf{D}$ of the excess demand function. Thus, the negative semidefiniteness of a Jacobian has been a key element in the notion of static stability in economic theory since John Hicks (1939) proposed it.

3.2 Dynamic stability of $\mathbf{V}^{\mathbf{F}}$

Mathematically, stability of an equilibrium in an autonomous dynamic system simply means convergence of a solution path to the equilibrium from an arbitrary point—in the entire space for global stability and in some neighborhood for local stability.[25] In mathematical study of dynamical systems, a Lyapunov function is a common tool to prove stability of equilibrium. In this paper, we will use the following version of the Lyapunov stability theorem for a differential inclusion.[26]

Theorem 1 (Lyapunov stability theorem, extended from Zusai, 2018, Theorem 7)**.** *Let X^* be a non-empty closed set in a compact metric space $\mathcal{X}$ with a tangent space $T\mathcal{X}$, and X' be a neighborhood of X^*. Suppose that continuous function $W : \mathcal{X} \to \mathbb{R}$ and lower semicontinuous function $\tilde{W} : \mathcal{X} \to \mathbb{R}$ satisfy (a) $W(\mathbf{x}) \geq 0$ and $\tilde{W}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in X'$ and (b) $\operatorname{cl} X' \cap W^{-1}(0) = X' \cap \tilde{W}^{-1}(0) = X^*$. In addition, assume that W is Lipschitz continuous in $\mathbf{x} \in X'$. i) If a differential inclusion $\boldsymbol{\mathcal{V}} : \mathcal{X} \to T\mathcal{X}$ satisfies*

$$\frac{d}{d\mathbf{x}}W(\mathbf{x})\dot{\mathbf{x}} \leq \underbrace{\tilde{W}(\mathbf{x})}_{\leq 0 \text{ by (a)}} \qquad \text{for any } \dot{\mathbf{x}} \in \boldsymbol{\mathcal{V}}(\mathbf{x}) \tag{5}$$

whenever W is differentiable at $\mathbf{x} \in X'$, then X^* is asymptotically stable under $\boldsymbol{\mathcal{V}}$. ii) If X'

of $\mathbf{x}^*$. We may extend local static and dynamic stability to a set of connected Nash equilibria.

[25] See Sandholm (2010b, §7.A) for formal definitions of mathematical stability concepts, such as asymptotic stability, Lyapunov stability and attracting, especially for a differential inclusion.

[26] See Supplement S2.2 for the proof and the difference from Zusai (2018, Theorem 7) and standard textbook versions of Lyapunov stability theorem for differential equations and differential inclusions.

is forward invariant, i.e., every Carathéodory solution path $\{\mathbf{x}_t\}$ *starting from* X' *at time* 0 *remains in* X' *for all moments of time* $t \in [0, \infty)$*, then* X' *is a basin of attraction to* X^**.*

We call W a *Lyapunov function* and $\tilde{W}$ a *decaying rate function*. Note that we allow multiplicity of transition vectors, while requiring functions W and $\tilde{W}$ to be well defined (the uniqueness of the values) as functions of the state variable $\mathbf{x}$, independently of the choice of transition vector $\dot{\mathbf{x}}$ from $\mathcal{V}(\mathbf{x})$. In the preceding literature, different Lyapunov functions are used for different dynamics to prove global stability of Nash equilibria in a contractive game; see Figure 2 in Section 4.

3.3 δ-passivity of V as a bridge

In this paper, we try to identify commonality over different evolutionary dynamics $\mathbf{V}$ that yields dynamic stability of $\mathbf{V}^{\mathbf{F}}$ when combined with static stability of $\mathbf{V}^{\mathbf{F}}$. Based on the separatists' viewpoint of game dynamics, Fox and Shamma (2013) define δ-passivity as a mathematical property of the evolutionary dynamic $\mathbf{V}$.[27]

Definition 2 (δ-passivity, a slightly extended from Fox and Shamma, 2013)**.** Evolutionary dynamic $V : \mathcal{X} \times \mathbb{R}^A \rightrightarrows T\mathcal{X}$ is δ-**passive**, if there exists a *pre-Lyapunov function* $L : \mathcal{X} \times \mathbb{R}^A \to \mathbb{R}_+$ and a *pre-decaying rate function* $\tilde{L} : \mathcal{X} \times \mathbb{R}^A \to \mathbb{R}_-$ s.t., at any $\mathbf{x} \in \mathcal{X}$ and $\boldsymbol{\pi} \in \mathbb{R}^A$,

$$\frac{\partial L}{\partial \mathbf{x}}(\mathbf{x})[\boldsymbol{\pi}]\Delta\mathbf{x} + \frac{\partial L}{\partial \boldsymbol{\pi}}(\mathbf{x})[\boldsymbol{\pi}]\Delta\boldsymbol{\pi} \leq \Delta\mathbf{x} \cdot \Delta\boldsymbol{\pi} + \tilde{L}(\mathbf{x})[\boldsymbol{\pi}] \qquad \text{for any } \Delta\boldsymbol{\pi} \in \mathbb{R}^A \text{ and } \Delta\mathbf{x} \in \mathbf{V}(\mathbf{x})[\boldsymbol{\pi}].$$

In a game $\mathbf{F}$, payoff vector $\boldsymbol{\pi}$ is always determined from the current population state $\mathbf{x}$. Thus, the pre-Lyapunov function L reduces to a function of $\mathbf{x}$ given by $L^{\mathbf{F}}(\mathbf{x}) := L(\mathbf{x})[\mathbf{F}(\mathbf{x})]$; similarly, $\tilde{L}^{\mathbf{F}}(\mathbf{x}) := \tilde{L}(\mathbf{x})[\mathbf{F}(\mathbf{x})]$. If the game $\mathbf{F}$ satisfies static stability at $\mathbf{x}$, then

$$\dot{\mathbf{x}} \cdot \dot{\boldsymbol{\pi}} = \dot{\mathbf{x}} \cdot \frac{d\mathbf{F}}{d\mathbf{x}}(\mathbf{x})\dot{\mathbf{x}} \leq 0 \text{ with } \dot{\boldsymbol{\pi}} = \frac{d\mathbf{F}}{d\mathbf{x}}(\mathbf{x})\dot{\mathbf{x}} \in \mathbb{R}^A \qquad \text{for any } \dot{\mathbf{x}} \in T\mathcal{X}.$$

[27] The δ refers to differentiation, i.e., the time derivatives of state and costate variables. Fox and Shamma (2013) add it to the term in order to distinguish from passivity of the current values of these variables.

Since our version of the Lyapunov stability theorem requires the explicit specification of a lower semicontinuous function $\tilde{W}$ as a decaying rate function to cover a differential inclusion, the definition of δ-passivity differs from Fox and Shamma (2013) by including $\tilde{L}$ in the RHS of the defining equation.

Thus, δ-passivity of $\mathbf{V}$ implies (5) for $\mathbf{V}^{\mathbf{F}}$:

$$\dot{L}^{\mathbf{F}}(\mathbf{x}) = \frac{\partial L}{\partial \mathbf{x}}(\mathbf{x})[\mathbf{F}(\mathbf{x})]\dot{\mathbf{x}} + \frac{\partial L}{\partial \boldsymbol{\pi}}(\mathbf{x})[\mathbf{F}(\mathbf{x})]\frac{d\mathbf{F}}{d\mathbf{x}}(\mathbf{x})\dot{\mathbf{x}} \leq \dot{\mathbf{x}} \cdot \frac{d\mathbf{F}}{d\mathbf{x}}(\mathbf{x})\dot{\mathbf{x}} + \tilde{L}^{\mathbf{F}}(\mathbf{x}) \leq \tilde{L}^{\mathbf{F}}(\mathbf{x}).$$

This suggests that we can use $L^{\mathbf{F}}$ as a Lyapunov function for dynamic $\mathbf{V}^{\mathbf{F}}$, with $\tilde{L}^{\mathbf{F}}$ as a decaying rate function to apply Theorem 1.

δ-passivity points out exactly the *mathematical* property that should be satisfied for an evolutionary dynamic $\mathbf{V}$, *separately from a game* $\mathbf{F}$, in order to derive dynamic stability of $\mathbf{V}^{\mathbf{F}}$ from static stability of $\mathbf{F}$. However it does not still tell how to *find* a pre-Lyapunov function.

4 Main results

Now we have clarified our goal: find a pre-Lyapunov function for an economically reasonable dynamic, based on economic intuition. For this, we need an *exact* and intuitive measure that tells whether or not agents should change their choices. In an economically reasonable dynamic, an agent chooses to switch the action at a revision opportunity *whenever* the net gain is positive; the agent chooses to keep the current action *if it is negative*. Thus, the net gain is an exact measure to assess when an agent chooses to switch the strategy and *when not*. Below we make this idea concrete and construct the pre-Lyapunov function from net gains.

4.1 Individual ex-ante net gains

Defining net gain from revisions

Imagine an agent who receives a revision opportunity and draws available strategy set $\mathcal{A}'_a$ and switching cost q from their distributions $\mathbb{P}_{\mathcal{A}a}$ and $\mathbb{P}_Q$. If the agent switches an action, the agent should choose the best available action in $\mathcal{A}'_a$; the net gain from this switch is $\pi_*(\mathcal{A}'_a) - \pi_a - q$. Before the realization of $\mathcal{A}'_a$ and q, the

individual ex-ante (first-order) net gain for an action-a player is represented by[28]

$$g_{a*}[\boldsymbol{\pi}] \coloneqq \sum_{\mathcal{A}'_a \subset \mathcal{A}\setminus\{a\}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a)\mathbb{E}_Q[\pi_*(\mathcal{A}'_a) - \pi_a - q]_+. \tag{6}$$

This represents the remaining room to improve the payoff for a player who currently takes action a. The net gain $\pi_*(\mathcal{A}'_a) - \pi_a - q$ from a switch is counted into the expected value if and only if the agent chooses the switch, i.e., the net gain is positive. Let $\mathbf{g}_*[\boldsymbol{\pi}] \coloneqq (g_{a*}[\boldsymbol{\pi}])_{a\in\mathcal{A}} \in \mathbb{R}^A$.

Lemma 1. *Consider an economically reasonable dynamic* $\mathbf{V}$*. Then, the ex-ante first-order gain function* $\mathbf{g}_* : \mathbb{R}^A \to \mathbb{R}^A$ *satisfies the following properties.*

g0 *i)* $g_{a*}[\boldsymbol{\pi}] \geq 0$ *for any* $a \in \mathcal{A}$ *and* $\boldsymbol{\pi} \in \mathbb{R}^A$*. ii)* $g_{a*}[\boldsymbol{\pi}] = 0$ *if and only if* $a \in b_*[\boldsymbol{\pi}]$*.*[29]

g1 *Consider arbitrary two actions* $a, b \in \mathcal{A}$*; assume* $\pi_a \leq \pi_b$*. Then, i)* $g_{b*}[\boldsymbol{\pi}] \leq g_{a*}[\boldsymbol{\pi}]$*. ii) The inequality is strict if* $\pi_a < \pi_b$*.*

g2 *For each* $a \in \mathcal{A}$*, function* $g_{a*} : \mathbb{R}^A \to \mathbb{R}$ *is Lipschitz continuous, and thus differentiable almost everywhere in* $\mathbb{R}^A$*. If it is differentiable at* $\boldsymbol{\pi} \in \mathbb{R}^A$*,*

$$\frac{\partial g_{a*}}{\partial \boldsymbol{\pi}}[\boldsymbol{\pi}]\Delta\boldsymbol{\pi} = \mathbf{z}_a \cdot \Delta\boldsymbol{\pi} \quad \text{for any } \Delta\boldsymbol{\pi} \in \mathbb{R}^A, \mathbf{z}_a \in \mathbf{V}_a[\boldsymbol{\pi}]. \tag{7}$$

About g0, it is immediate from the definition (6) to confirm i) and the "if" part of ii). The "only-if" part relies on Assumption A1-i), which implies that any strategy must be available with some positive probability.

Property g1 states that the better the status-quo (current) action is, the less the ex-ante gain is. So, by choosing the best among available actions $b \in b_*[\boldsymbol{\pi}](\mathcal{A}'_a)$, an agent shrinks the remaining, unexploited gain to the smallest among those of available actions, i.e., $g_{**}[\boldsymbol{\pi}](\mathcal{A}'_a) \coloneqq \min_{b\in\mathcal{A}'_a} g_{b*}[\boldsymbol{\pi}]$. For this property, we observe from (6) that the status-quo action a affects the ex-ante net gain through two factors. The most obvious is through the default payoff $-\pi_\cdot$ in calculation of the realized

[28]The block parentheses with a positive sign $[\cdot]_+$ indicates the non-negative part of the argument: i.e, $[z]_+ = z$ if $z > 0$ and $[z]_+ = 0$ if $z \leq 0$. $\mathbb{E}_Q$ is the expected value operator: $\mathbb{E}_Q f(q) = \int_{\mathbb{R}} f(q) d\mathbb{P}_Q(q)$ for integrable function $f : \mathbb{R} \to \mathbb{R}$. Note that this is well defined (the value of RHS is uniquely determined without choosing a particular $p_Q(\pi_*(\mathcal{A}'_a) - \pi_a)$ from $\mathcal{Q}(\pi_*(\mathcal{A}'_a) - \pi_a)$) without assuming continuity of $\mathbb{P}_Q$.

[29]For property g0, we do not need Assumption A1-ii).

net gain. The less obvious is through the availability of other actions in calculation of probability $\mathbb{P}_{\mathcal{A}\cdot}(\mathcal{A}'_\cdot)$. AssumptionA1-ii) eliminates this secondary factor; see Supplement S3.1 Hence, the only first factor matters, which leads us to g1.

To understand property g2, imagine a change in payoff vector by $\Delta\boldsymbol{\pi}$. Then, the net gain $\pi_*(\mathcal{A}'_a) - \pi_a - q$ changes by $\Delta\pi_*(\mathcal{A}'_a) - \Delta\pi_a$ regardless of q. The resulting change in $g_{a*}[\boldsymbol{\pi}]$ is linearly approximated as[30]

$$\frac{\partial g_{a*}}{\partial \boldsymbol{\pi}}[\boldsymbol{\pi}]\Delta\boldsymbol{\pi} = \sum_{\mathcal{A}'_a \subset \mathcal{A}\setminus\{a\}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) Q(\Delta\pi_*(\mathcal{A}'_a) - \Delta\pi_a) \cdot (\Delta\pi_*(\mathcal{A}'_a) - \Delta\pi_a).$$

This is the expected sum of the change in the net gain $\Delta\pi_*(\mathcal{A}'_a) - \Delta\pi_a$ weighted with the current probability that a switch from a is chosen, i.e., probability $\mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a)Q(\pi_*(\mathcal{A}'_a) - \pi_a)$, given available action set $\mathcal{A}'_a$; this weighted sum is simply $\mathbf{z}_a \cdot \Delta\boldsymbol{\pi}$.[31] Note that this does not depend on a choice of transition vector $\mathbf{z}_a$ from $\mathbf{V}_a[\boldsymbol{\pi}]$. It is crucial for property g2 that *zero* net gain sets a threshold between switching to a new action and keeping the current action. This makes a contrast with gross gain, as we argue in Section 6.1.

The second-order net gains

As a candidate for the decaying rate function, we define the **individual ex-ante second-order net gain** for an action-a player as

$$h_{a*}[\boldsymbol{\pi}] \coloneqq \sum_{\mathcal{A}'_a \subset \mathcal{A}\setminus\{a\}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) Q(\pi_*(\mathcal{A}'_a) - \pi_a)(g_{**}[\boldsymbol{\pi}](\mathcal{A}'_a) - g_{a*}[\boldsymbol{\pi}]). \tag{8}$$

The term $\mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a)Q(\pi_*(\mathcal{A}'_a) - \pi_a)$ is the (greatest) probability[32] that, upon the receipt of the first revision opportunity, the agent receives available action set $\mathcal{A}'_a$ and chooses to switch from the current action a. Property g1 implies that this switch changes the agent's first-order net gain from $g_{a*}[\boldsymbol{\pi}]$ to $g_{**}[\boldsymbol{\pi}](\mathcal{A}'_a)$. Hence $h_{a*}[\boldsymbol{\pi}]$

[30]To obtain this equation mathematically, notice that the derivative of $\mathbb{E}_Q[\check{\pi} - q]_+ = \int_{q=0}^{\check{\pi}} (\check{\pi} - q) d\mathbb{P}_Q(q)$ with respect to $\check{\pi}$ is $\int_{q=0}^{\check{\pi}} 1 \cdot d\mathbb{P}_Q(q) = \mathbb{P}_Q(q \le \check{\pi}) = Q(\check{\pi})$.

[31]When g_{a*} is differentiable at $\boldsymbol{\pi}$, then $\mathbb{P}_Q$ must be continuous at $\pi_*(\mathcal{A}'_a) - \pi_a$ of all $\mathcal{A}'_a \subset \mathcal{A} \setminus \{a\}$ for all $a \in \mathcal{A}$.

[32]If $\mathbb{P}_Q$ is not continuous, $Q(\pi_*(\mathcal{A}'_a) - \pi_a)$ is only the greatest possible probability of switch as the switching probability can be any value in the range $\mathcal{Q}(\pi_*(\mathcal{A}'_a) - \pi_a)$. Yet, we keep this definition to define h_{a*} as a (single-valued) function of $\boldsymbol{\pi}$. So h_{a*} is lower semicontinuous thanks to upper semicontinuity of cumulative distribution function Q.

is the expected change in the first-order gain upon the first revision opportunity, as stated in property gh below.

If $Q(\pi_*(\mathcal{A}'_a) - \pi_a) > 0$, i.e., an agent chooses switching from a with a positive probability, then it must be the case that $\pi_*(\mathcal{A}'_a) \geq \pi_a$ and thus, by property g1, $g_{**}[\boldsymbol{\pi}](\mathcal{A}'_a) \leq g_{a*}[\boldsymbol{\pi}]$; so we obtain property h1-i). Further, if a is not optimal, then there must be a better action, which must be available with some positive probability by Assumption A1-i). The above argument suggests that the second-order gain must be negative in such a case. The contrapositive is property h1-ii).

Lemma 2. *Assume that the ex-ante gain function $\mathbf{g}_* : \mathbb{R}^A \to \mathbb{R}^A$ satisfies property g1.*[33] *Then, the ex-ante second-order gain function $\mathbf{h}_* = (h_{a*})_{a\in\mathcal{A}} : \mathbb{R}^A \to \mathbb{R}^A$ satisfies the following properties.*

h *h_{a*} is lower semicontinuous. i) $h_{a*}[\boldsymbol{\pi}] \leq 0$ for any $a \in \mathcal{A}$ and $\boldsymbol{\pi} \in \mathbb{R}^A$. ii) $h_{a*}[\boldsymbol{\pi}] = 0$ if and only if $a \in b_*[\boldsymbol{\pi}]$.*

gh *For any $a \in \mathcal{A}$ and $\boldsymbol{\pi} \in \mathbb{R}^A$, if g_{a*} is differentiable at $\boldsymbol{\pi}$, we have*

$$h_{a*}[\boldsymbol{\pi}] = \mathbf{z}_a \cdot \mathbf{g}_*[\boldsymbol{\pi}] \qquad \text{for any } \mathbf{z}_a \in \mathbf{V}_a[\boldsymbol{\pi}].$$

If it is not differentiable, this holds with a weak inequality $\geq$.

4.2 Aggregate net gain

To analyze the overall social dynamic, we aggregate the ex-ante first-order and second-order gains over the whole population. We define the **aggregate first/second-order gain functions** $G, H : \Delta^{\mathcal{A}} \to \mathbb{R}$ as

$$G(\mathbf{x})[\boldsymbol{\pi}] := \mathbf{x} \cdot \mathbf{g}_*[\boldsymbol{\pi}] \quad \text{and} \quad H(\mathbf{x})[\boldsymbol{\pi}] := \mathbf{x} \cdot \mathbf{h}_*[\boldsymbol{\pi}].$$

Theorem 2 (Properties of gains under economic reasonable dynamics)**.** *Consider an economically reasonable dynamic $\mathbf{V}$. Then, the aggregate ex-ante gain function $G : \mathcal{X} \times \mathbb{R}^A \to \mathbb{R}$ and the aggregate ex-ante second-order gain function $H : \mathcal{X} \times \mathbb{R}^A \to \mathbb{R}^A$ satisfy the following properties.*

[33] No other assumptions or properties are needed. Once property g1 is satisfied, any assumption is not needed.

G *G is Lipschitz continuous (and thus differentiable almost everywhere). Besides, i) $G(\mathbf{x})[\boldsymbol{\pi}] \geq 0$ for any $\mathbf{x} \in \Delta^{\mathcal{A}}, \boldsymbol{\pi} \in \mathbb{R}^A$; and, ii) $G(\mathbf{x})[\boldsymbol{\pi}] = 0$ if and only if $\mathbf{x} \in \Delta^{\mathcal{A}}(b_*[\boldsymbol{\pi}])$.*

H *H is lower semicontinuous. Besides, i) $H(\mathbf{x})[\boldsymbol{\pi}] \leq 0$ for any $\mathbf{x} \in \Delta^{\mathcal{A}}, \boldsymbol{\pi} \in \mathbb{R}^A$; and, ii) $H(\mathbf{x})[\boldsymbol{\pi}] = 0$ if and only if $\mathbf{x} \in \Delta^{\mathcal{A}}(b_*[\boldsymbol{\pi}])$.*

GH *Whenever G is differentiable at $\mathbf{x} \in \Delta^{\mathcal{A}}, \boldsymbol{\pi} \in \mathbb{R}^A$, we have*[34]

$$i)\ \frac{\partial G}{\partial \mathbf{x}}(\mathbf{x})[\boldsymbol{\pi}]\Delta\mathbf{x} = H(\mathbf{x})[\boldsymbol{\pi}] \qquad ii)\ \frac{\partial G}{\partial \boldsymbol{\pi}}(\mathbf{x})[\boldsymbol{\pi}]\Delta\boldsymbol{\pi} = \Delta\mathbf{x} \cdot \Delta\boldsymbol{\pi}$$

for any $\Delta\mathbf{x} \in \mathbf{V}(\mathbf{x})[\boldsymbol{\pi}], \Delta\boldsymbol{\pi} \in \mathbb{R}^A$.

Part i) in each of properties G and H confirms that the aggregate net gain G is always non-negative and the aggregate second-order net gain H is always non-positive. Part ii) states that they are both zero if and only if (almost) all agents are taking the optimal strategies given the current payoff vector. In property GH, part i) comes from linearity of G in $\mathbf{x}$ and property gh of the second-order gain, since

$$\frac{\partial G}{\partial \mathbf{x}}(\mathbf{x})[\boldsymbol{\pi}]\Delta\mathbf{x} = \mathbf{g}_*[\boldsymbol{\pi}] \cdot \Delta\mathbf{x} = \mathbf{g}_*[\boldsymbol{\pi}] \cdot \left(\sum_{a \in \mathcal{A}} x_a \mathbf{z}_a \right) = \sum_{a \in \mathcal{A}} x_a h_{a*}[\boldsymbol{\pi}] = H(\mathbf{x})[\boldsymbol{\pi}],$$

with any $\mathbf{z}_a \in \mathbf{V}_a[\boldsymbol{\pi}]$ for each $a \in \mathcal{A}$. For linearity of G in $\boldsymbol{\pi}$, notice that Assumption A0 assures invariance of $g_{a*}[\boldsymbol{\pi}]$ under transition of $\mathbf{x}$. Part ii) is induced from property g2 of $g_{a*}[\boldsymbol{\pi}]$, as

$$\frac{\partial G}{\partial \boldsymbol{\pi}}(\mathbf{x})[\boldsymbol{\pi}]\Delta\boldsymbol{\pi} = \sum_{a \in \mathcal{A}} x_a \frac{\partial g_{a*}}{\partial \boldsymbol{\pi}}(\mathbf{x})[\boldsymbol{\pi}]\Delta\boldsymbol{\pi} = \sum_{a \in \mathcal{A}} x_a \mathbf{z}_a \cdot \Delta\boldsymbol{\pi} = \Delta\mathbf{x} \cdot \Delta\boldsymbol{\pi}$$

with any $\mathbf{z}_a \in \mathbf{V}_a[\boldsymbol{\pi}]$ for each $a \in \mathcal{A}$.

If the payoff vector never changes from $\boldsymbol{\pi}^0$, i.e., $\mathbf{F}(\mathbf{x}) \equiv \boldsymbol{\pi}^0$ for any $\mathbf{x}$, property GH-i) alone implies $\dot{G}(\mathbf{x})[\boldsymbol{\pi}^0] = \frac{\partial G}{\partial \mathbf{x}}(\mathbf{x})[\boldsymbol{\pi}^0]\dot{\mathbf{x}} = H(\mathbf{x})[\boldsymbol{\pi}^0]$. With property H, this suggests that the aggregate net gain must diminish toward zero under any economically reasonable dynamics. Without strategic dependency of the payoffs on state $\mathbf{x}$, this simply suggests a process in which economic agents gradually but steadily exploit opportunities to make greater payoffs until they completely reach optimal choices given the constant payoff vector.

[34] Note that Assumption A0 is needed only for property GH in this theorem.

When we think of a "game", the payoff vector changes with state $\mathbf{x}$. Thanks to properties GH-i) and ii), an economically reasonable dynamic satisfies δ-passivity (Definition 2) with function G as the pre-Lyapunov function and H as the pre-decaying rate function:

$$\frac{\partial G}{\partial \mathbf{x}}(\mathbf{x})[\boldsymbol{\pi}]\Delta\mathbf{x} + \frac{\partial G}{\partial \boldsymbol{\pi}}(\mathbf{x})[\boldsymbol{\pi}]\Delta\boldsymbol{\pi} = H(\mathbf{x})[\boldsymbol{\pi}] + \Delta\mathbf{x}\cdot\Delta\boldsymbol{\pi} \qquad \text{for any } \Delta\mathbf{x} \in \mathbf{V}(\mathbf{x})[\boldsymbol{\pi}], \Delta\boldsymbol{\pi} \in \mathbb{R}^A.$$

Corollary 1. *Any economically reasonable dynamic is δ-passive.*

4.3 From static stability to dynamic stability

Now we embed population game $\mathbf{F}$ into evolutionary dynamic $\mathbf{V}$ in order to examine dynamic stability of the induced game dynamic $\mathbf{V}^{\mathbf{F}}$. Let $G^{\mathbf{F}}(\mathbf{x}) := G(\mathbf{x})[\mathbf{F}(\mathbf{x})]$ and $H^{\mathbf{F}}(\mathbf{x}) := H(\mathbf{x})[\mathbf{F}(\mathbf{x})]$. Combined with static stability of $\mathbf{F}$, δ-passivity of $\mathbf{V}$ implies that $G^{\mathbf{F}}$ works as a Lyapunov function with $H^{\mathbf{F}}$ as a decaying rate function to prove asymptotic stability of the equilibrium (set). Our construction of the gain function G is consistent with the Lyapunov function that has been found in the preceding papers on each of major dynamics: see Figure 2.

Theorem 3 (Dynamic stability from static stability through net gains)**.** *Consider an evolutionary dynamic $\mathbf{V}$ that generates the total ex-ante first/second-order gain functions G, H satisfying properties G, H, and GH in Theorem 2. Then, static stability of equilibrium in $\mathbf{F}$ (i.e., negative semidefiniteness of $D\mathbf{F}$) implies dynamic stability under $\mathbf{V}^{\mathbf{F}}$. The aggregate net gain $G^{\mathbf{F}}$ monotonically decreases toward zero. Specifically, we have the following.*

i) Global static stability of $\mathbf{F}$ implies global asymptotic stability of $\mathrm{NE}(\mathbf{F})$ under the induced game dynamic $\mathbf{V}^{\mathbf{F}}$.

ii) In an arbitrary population game $\mathbf{F}$, let $\mathbf{x}^* \in \mathcal{X}$ be an isolated Nash equilibrium. (a) Local static stability of $\mathbf{F}$ at $\mathbf{x}^*$ implies (local) asymptotic stability of $\mathbf{x}^*$ under the induced game dynamic $\mathbf{V}^{\mathbf{F}}$. (b) Consider a case that $D\mathbf{F}(\mathbf{x}^*)$ is not negative semidefinite. If there is indeed a transition vector $\mathbf{z} \in \mathbf{V}^{\mathbf{F}}(\mathbf{x}^*)$ such that $\mathbf{z}\cdot D\mathbf{F}(\mathbf{x}^*)\mathbf{z} > 0$, then $\mathbf{x}^*$ is not Lyapunov stable.

Recall that negative semidefiniteness of $D\mathbf{F}$ represents static stability of a game and characterizes a contractive game and an interior ESS, as argued in Section 3. So this theorem reduces to the following.

Economically reasonable dynamics in the basic framework (this paper)

$\mathbb{P}_{\mathcal{A}a}$: Assumptions A1-i),A1-ii); Q: Assumption Q1.

- $\mathbf{V}(\mathbf{x})[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a \sum_{\mathcal{A}'_a\subset\mathcal{A}\setminus\{a\}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) Q(\pi_*(\mathcal{A}'_a) - \pi_a)(\Delta^{\mathcal{A}}(b_*[\boldsymbol{\pi}](\mathcal{A}'_a)) - \mathbf{e}_a)$.
- $G(\mathbf{x})[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a \sum_{\mathcal{A}'_a\subset\mathcal{A}\setminus\{a\}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) \mathbb{E}_Q[\pi_*(\mathcal{A}'_a) - \pi_a - q]_+$.
- $H(\mathbf{x})[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a \sum_{\mathcal{A}'_a\subset\mathcal{A}\setminus\{a\}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) Q(\pi_*(\mathcal{A}'_a) - \pi_a)(g_{**}[\boldsymbol{\pi}](\mathcal{A}'_a) - g_{a*}[\boldsymbol{\pi}])$.

Tempered BRDs (Zusai, 2018)

$\mathbb{P}_{\mathcal{A}a}(A\setminus\{a\}) = 1$; Q: any function that meets Assumption Q1.

- $\mathbf{V}(\mathbf{x})[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a Q(\pi_* - \pi_a)(\Delta^{\mathcal{A}}(b_*[\boldsymbol{\pi}]) - \mathbf{e}_a)$.
- $G(\mathbf{x})[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a \mathbb{E}_Q[\pi_* - \pi_a - q]_+$.
- $H(\mathbf{x})[\boldsymbol{\pi}] = -\sum_{a\in\mathcal{A}} x_a Q(\pi_* - \pi_a)\mathbb{E}_Q[\pi_* - \pi_a - q]_+$.

Standard BRD (Hofbauer, 1995)

$\mathbb{P}_{\mathcal{A}a}(A\setminus\{a\}) = 1$; $Q(q) = 1$ for any $q > 0$.

- $\mathbf{V}(\mathbf{x})[\boldsymbol{\pi}] = \Delta^{\mathcal{A}}(b_*[\boldsymbol{\pi}]) - \mathbf{x}$.
- $G(\mathbf{x})[\boldsymbol{\pi}] = \pi_* - \mathbf{x}\cdot\boldsymbol{\pi} = \sum_{a\in\mathcal{A}} x_a[\pi_* - \pi_a]_+$.
- $H(\mathbf{x})[\boldsymbol{\pi}] = -\sum_{a\in\mathcal{A}} x_a[\pi_* - \pi_a]_+ = -G(\mathbf{x})[\boldsymbol{\pi}]$.

Figure 2: Major dynamics captured under the basic framework. A box containing another box suggests that the class of dynamics represented in a larger box includes the dynamics in a smaller box. The parts in red are $\mathbf{V}_a(\mathbf{x})[\boldsymbol{\pi}]$, $g_{a*}(\mathbf{x})[\boldsymbol{\pi}]$ and $h_{a*}(\mathbf{x})[\boldsymbol{\pi}]$. (To match G with the formula of the Lyapunov functions in the literature, notice $\mathbb{E}_Q[\bar{q} - q]_+ = \int_0^{\bar{q}} Q(q)dq$ provided that $Q(0) = 0$ and Q is continuous over $\mathbb{R}$.) The reference on each dynamic is a preceeding paper that presents the Lyapunov function for the dynamic, though they may present only forms like $\mathbf{V}^{\mathbf{F}}$ and $G^{\mathbf{F}}$ and mostly do not pay attention to H. (Anyway, as far as the author finds from these papers, none of them give economic interpretations of G as a gain.) Hofbauer (1995, 2001) initially proposed those Lyapunov functions for a game with an interior ESS.

Economically reasonable dynamic in the basic frame, continued.

Pairwise comparison dynamics (Hofbauer and Sandholm, 2009)

$\mathbb{P}_{\mathcal{A}a}(\{b\}) = 1/(A-1)$ for each $b \neq a$; Q: any function that meets Assumption Q1.

- $\mathbf{V}(\mathbf{x})[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a \sum_{b\neq a} \frac{Q(\pi_b - \pi_a)}{A-1}(\mathbf{e}_b - \mathbf{e}_a)$.
- $G(\mathbf{x})[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a \sum_{b\neq a} \frac{1}{A-1}\mathbb{E}_Q[\pi_b - \pi_a - q]_+$.
- $H(\mathbf{x})[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a \sum_{b\neq a} \frac{Q(\pi_b - \pi_a)}{(A-1)^2}\left(\sum_{c\neq b}\mathbb{E}_Q[\pi_c - \pi_b - q]_+ - \sum_{c\neq a}\mathbb{E}_Q[\pi_c - \pi_a - q]_+\right)$.

Smith dynamic (Smith, 1984)

$\mathbb{P}_{\mathcal{A}a}(\{b\}) = 1/(A-1)$ for each $b \neq a$; $Q(q) = [q]_+$.

- $\mathbf{V}(\mathbf{x})[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a \sum_{b\neq a} \frac{[\pi_b - \pi_a]_+}{A-1}(\mathbf{e}_b - \mathbf{e}_a)$.
- $G(\mathbf{x})[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a \sum_{b\neq a} \frac{1}{2(A-1)}[\pi_b - \pi_a]^2$.
- $H(\mathbf{x})[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a \sum_{b\neq a} \frac{[\pi_b - \pi_a]_+}{(A-1)^2}\left(\sum_{c\neq b}\frac{[\pi_c - \pi_b]_+^2}{2} - \sum_{c\neq a}\frac{[\pi_c - \pi_a]_+^2}{2}\right)$.

Figure 2: (Continued)

Corollary 2 (Equilibrium stability under economic reasonable dynamics)**.** *Consider an economically reasonable dynamic. In a contractive game, the set of Nash equilibria is globally asymptotically stable. An interior ESS is (locally) asymptotically stable.*

As we argued, δ-passivity is a mathematical ground to establish dynamic stability of contractive games. By generalizing the contraction property of a (static) game $\mathbf{F} : \mathbf{x} \mapsto \boldsymbol{\pi}$ to δ-*anti*-passivity of a dynamic payoff generating system $\{\mathbf{x}_t\}_{t\in\mathbb{R}_+} \mapsto \{\boldsymbol{\pi}_t\}_{t\in\mathbb{R}_+}$, Fox and Shamma (2013) extend dynamic stability in a contractive game under δ-passive evolutionary dynamics to a setting where agents base their decisions on some variables that summarize past payoffs and give a prediction.

5 Extensions beyond the basic set-up

5.1 Economic reasonability and gains in other major dynamics

We would emphasize that the "generality" of our approach would lie in the idea itself—to be summarized in Section 7; the analysis based on the "basic model" is made to convince readers of mathematical validity of our approach and to present a concrete way to construct the gain functions and prove crucial properties, as well as to set a boundary with what are excluded from our approach as argued in Section 6.

The essence of our approach is that static stability is proven to be a sufficient condition for dynamic stability as long as we can justify the dynamic in an economically reasonable way, i.e., based on optimization by revealing additional costs or constraints that prevent agents from exact optimization. This is indeed what has been done for smoothed best response dynamics, where agents follow a kind of quantal response rules such as logit choice rules. According to Hofbauer and Sandholm (2002), there are two ways to make it economically reasonable. First, each agent is considered as facing heterogeneous payoff perturbations while the agent chooses the exact optimal choice given the *perturbed* payoff vector. The perturbed BRD emerges as the aggregation of such heterogeneous exact BRD. We cover heterogeneous dynamics in Section 5.3 as an extension of the basic model to games with finitely many distinctive populations. Second, an agent chooses mixed strategy $\mathbf{y} \in \Delta^{\mathcal{A}}$ and there is a control cost $V(\mathbf{y})$ to prevent the agent from taking a pure strategy. The control cost perturbs the agent's optimal strategy. We can interpret the known Lyapunov function for perturbed BRDs in Hofbauer and Sandholm (2007,

Theorem 3.1) as the aggregate net gain function after counting the control costs and confirm that it satisfies properties G, H and GH.

Economic reasonability of excess payoff dynamics may not be obvious. In those dynamics, an agent's switching rate to a new action is an increasing function of excess payoff, i.e., the relative payoff of the action compared with the *average payoff in the population* $\boldsymbol{\pi} \cdot \mathbf{x}$. We can regard the population average strategy, i.e., $\mathbf{x}$, as the mixed-strategy status quo for the revising agent: if an agent chooses the status quo to avoid the payment of a switching cost, the new action is chosen randomly according to the current action distribution $\mathbf{x} \in \Delta^{\mathcal{A}}$ in the population; a revising agent can be interpreted as a newborn agent, replacing an old agent. Then, the relative payoff is justified as a determinant of the switching rate. See Supplement S4.1 for details. There we present the aggregate net gain function G and the accompanying decay rate function H, which is constructed to meet property GH-i).

5.2 Local stability on the boundary

While our local static stability condition would be natural when we imagine an interior Nash equilibrium, it may sound a bit too strong for a strict equilibrium on the boundary or, more weakly, a *quasi-strict equilibrium* $\mathbf{x}^*$, where all unsupported actions $U := \{u \in \mathcal{A} \mid x_u^* = 0\}$ are indeed strictly worse than supported actions $S := \{s \in \mathcal{A} \mid x_s^* > 0\}$. Economic intuition suggests that any small mutations to those worse actions should extinguish; even if those mutations may increase payoffs of those actions, the strict payoff domination of S over U remains locally. Then, we can relax local static stability condition by requiring negative semidefiniteness $\Delta\mathbf{x} \cdot D\mathbf{F}(\mathbf{x})\Delta\mathbf{x}$ only for deviations that keep the supported actions, i.e., $\Delta\mathbf{x} \in \mathbb{R}^{\mathcal{A}}_{S0} := \left\{\mathbf{z} \in \mathbb{R}^{\mathcal{A}}_0 \mid [z_u = 0 \text{ for any } u \in U]\right\}$.

This is indeed a weaker version of a regular ESS (Taylor and Jonker, 1978; Friedman, 1991; Sandholm, 2010a), which has been studied in the literature on dynamic stability of an ESS. Zusai (2019b) defines a weak regular ESS and prove its local stability under economically reasonable dynamics by generalizing an approach proposed by Sandholm (2010a) to construct a Lyapunov function for a regular ESS from the one for a contractive game; since it is crucial for a boundary equilibrium how close the current state is to the boundary, an additional term is added to account closeness to the boundary.

5.3 Heterogeneity

The model and all the propositions can be easily extended to a multi-population game in which (finitely many) different populations may have different payoff functions and/or different revision protocols, as long as each population's revision protocol is economically reasonable;[35] see Supplement S4.2.[36]

We can obtain the pre-Lyapunov function for a multi-population game just by adding up the net gain functions over all the populations; similarly, the decay rate function is obtained straightforwardly from summation of the corresponding functions over the populations. The separatist's view is behind this straightfoward extension. Choices in different populations affect evolution in other populations through playing a game $\mathbf{F}^{\mathcal{P}}$ on the same field.[37] However, the separation of an evolutionary dynamic $\mathbf{V}(\mathbf{x})[\boldsymbol{\pi}]$ from a game $\boldsymbol{\pi} = \mathbf{F}(\mathbf{x})$ allows us to put aside this indirect interdependence between populations.

The extension to the multi-population setting covers some interesting class of games. One is a *saddle game*, which is a generalization of zero-sum games and also recently related to the literature on robust equilibria under incomplete information by Nora and Uno (2014).[38] Another is an *aggregate game*, in which agents' payoffs depend only on the aggregate action distribution over the society. The aggregation represents unobservability of each agent's affiliating population when they play the game and their payoffs are assessed.

Sandholm (2002, 2005) proposes to extend the idea of Pigouvian tax to the evolutionary dynamic setting in order to achieve the maximal aggregate payoff at the limit state under negative externality, with the central planner knowing only the aggregate action distribution. To prove global convergence to the social optimum, the aggregate payoff functions is supposed to be concave; the Pigouvian tax makes it a

[35] It can be economically reasonable in the basic framework in this paper or in some modified ways as argued in Section 5.1. In particular, we can consider a hybrid dynamic where an agent may keep the current choice of an action as a status quo or may be replaced with a new agent. Even if a population's dynamic is not economically reasonable, we can include it as long as we can somehow find functions G and H that satisfy properties G, H and GH for this dynamic.

[36] Zusai (2019a) allows the number of types to be continuously many and provides the measure-theoretic definition of heterogeneous evolutionary dynamic with clarification of regularity conditions for the unique existence of a solution path; further, the paper verifies dynamic stability of the Nash equilibrium set in a potential game.

[37] Note that, because of this, even an uncoupled nonobservational dynamic cannot guarantee convergence to Nash equilibria in general (Hart and Mas-Colell, 2003).

[38] Evolutionary dynamics in this class of games can date back to Kose (1956) that consider computation of a saddle point problem through evolutionary algorithms.

potential function in a game with tax and thus concavity of the potential function implies global convergence. Since a contractive game is generalization of a concave potential game and removes the assumption of exact symmetry of externalities, our result suggests robustness of Sandholm's result.

6 Discussion: why not other ways?

6.1 Why not using the gross gain for a Lyapunov function?

For the net gain, we need to "invent" the switching cost q. One may think it artificial and wonder why we need it. Why can't we simply look at gross gains or payoff improvements? Under an economically reasonable dynamic, when an action-a player receives a revision opportunity, the agent draws available action set $\mathcal{A}'_a$ with probability $\mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a)$, chooses to switch the action (at least) with probability $Q_-(\pi_*(\mathcal{A}'_a) - \pi_a)$ and yields the gross gain $\pi_*(\mathcal{A}'_a) - \pi_a$. Thus, we can define the **individual ex-ante gross gain** $\boldsymbol{\gamma} = (\gamma_a)_{a\in\mathcal{A}} : \mathbb{R}^A \to \mathbb{R}^A$ as a function of $\boldsymbol{\pi}$:[39]

$$\gamma_a[\boldsymbol{\pi}] \coloneqq \sum_{\mathcal{A}'_a \subset \mathcal{A}\setminus\{a\}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) Q_-(\pi_*(\mathcal{A}'_a) - \pi_a) \left\{\pi_*(\mathcal{A}'_a) - \pi_a\right\}.$$

Given current strategy distribution $\mathbf{x}$, the individual gross gains over all agents aggregate to the **aggregate gross gain**

$$\Gamma(\mathbf{x})[\boldsymbol{\pi}] \coloneqq \mathbf{x} \cdot \boldsymbol{\gamma}[\boldsymbol{\pi}].$$

This Γ retains a property like property G: Γ is always non-negative, and attains zero if and only if (almost) all agents are taking the optimal actions given the current payoff vector.[40]

Why couldn't we use Γ^F as a Lyapunov function? Actually, it does not work: Γ^F does not monotonically decrease over time even in a strictly contractive game, as

[39] If $\mathbb{P}_Q$ has an atom at $\pi_*(\mathcal{A}'_a) - \pi_a$, the switching probability $Q_a(\mathcal{A}'_a)$ can be any probability in the range $\mathcal{Q}(\pi_*(\mathcal{A}'_a) - \pi_a)$. See footnote 32 for indeterminancy of the conditional switching rate. In the definition of $\gamma_a[\boldsymbol{\pi}]$, we take the lower bound $Q_-(\pi_*(\mathcal{A}'_a) - \pi_a)$ to retain lower semicontinuity of γ_a, which is needed to verify equilibrium stability of potential games by our Lyapunov stability theorem (Theorem 1).

[40] This property is the same as "positive correlation", which implies stability of Nash equilibrium set in a potential game (Sandholm, 2001).

illustrated in Example S11 in Supplement. Nonmonotonic change in the aggregate gross gain may happen especially if agents do not switch to the best response at each revision opportunity. With $\mathbf{x}$ fixed, imagine that there is a marginal change in the payoff vector $\Delta\boldsymbol{\pi}$. Differentiation of the aggregate gross gain yields

$$\frac{\partial \Gamma}{\partial \boldsymbol{\pi}}(\mathbf{x})[\boldsymbol{\pi}]\Delta\boldsymbol{\pi} = \underbrace{\sum_{a\in\mathcal{A}} x_a \sum_{\mathcal{A}'_a} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) Q_-(\check{\pi}_{a*}(\mathcal{A}'_a))\Delta\check{\pi}_{a*}(\mathcal{A}'_a)}_{\Delta\mathbf{x}\cdot\Delta\boldsymbol{\pi} \text{ for any } \Delta\mathbf{x}\in\mathbf{V}[\boldsymbol{\pi}](\mathbf{x})} + \sum_{a\in\mathcal{A}}\sum_{\mathcal{A}'_a} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) x_a \{Q'_-(\check{\pi}_{a*}(\mathcal{A}'_a))\Delta\check{\pi}_{a*}(\mathcal{A}'_a)\}\check{\pi}_{a*}(\mathcal{A}'_a),$$

where $\check{\pi}_{a*}(\mathcal{A}'_a) \coloneqq \pi_*(\mathcal{A}'_a) - \pi_a$ and $\Delta\check{\pi}_{a*}(\mathcal{A}'_a) \coloneqq \Delta\pi_*(\mathcal{A}'_a) - \Delta\pi_a$.

This equation suggests that the marginal change in the payoffs affects the aggregate gross gains via two ways. First, it directly changes the realized individual gross gain $\check{\pi}_{a*}(\mathcal{A}'_a)$ by $\Delta\check{\pi}_{a*}(\mathcal{A}'_a)$ for each agent who chooses to switch the action. As $x_a\mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a)Q_-(\check{\pi}_{a*}(\mathcal{A}'_a))$ is the mass (rate) of those switching agents (per unit time) among those who have been taking a, the first summation in the second line represents this direct effect. Since those switches aggregate to the transition of the action distribution $\Delta\mathbf{x} \in \mathbf{V}[\boldsymbol{\pi}](\mathbf{x})$, this is indeed $\Delta\mathbf{x}\cdot\Delta\boldsymbol{\pi}$. When the game $\mathbf{F}$ is embedded to the evolutionary dynamic, static stability of $\mathbf{F}$ requires $\Delta\boldsymbol{\pi} = D\mathbf{F}(\mathbf{x})\Delta\mathbf{x}$ to be negatively correlated with $\Delta\mathbf{x}$; hence, the first term is negative.

Second, $\Delta\boldsymbol{\pi}$ changes the conditional switching rate $Q_-(\check{\pi}_{a*}(\mathcal{A}'_a))$ by $Q'_-(\check{\pi}_{a*}(\mathcal{A}'_a))\Delta\check{\pi}_{a*}(\mathcal{A}'_a)$. This reflects changes in switching decisions for marginal agents who are on the threshold between choosing to switch and not. By changing from no switch to switch, each one of such marginal agents adds the individual gross gain $\check{\pi}_{a*}(\mathcal{A}'_a)$ to the aggregate gross gain Γ. Notice that $\check{\pi}_{a*}(\mathcal{A}'_a)$ may be significantly (not marginally) positive even for those marginal agents, since the *gross* gain does not count switching costs; it is not an exact determinant for switching decision unlike the net gain.

While Q in the first term accounts for $\dot{\mathbf{x}}$, Q' in the second term accounts for the change in $\dot{\mathbf{x}}$, i.e, $\ddot{\mathbf{x}}$. Thus, the direction of the change in the aggregate gross gain cannot be determined without knowing a higher-order time derivative $\ddot{\mathbf{x}}$, which depends on specifications of dynamics such as Q' and $\mathbb{P}_{\mathcal{A}a}$ beyond economic reasonability. Thus, we cannot expect a sufficient condition on $\mathbf{F}$ alone (another kind of static stability) to guarantee monotonic decrease in Γ and dynamic stability of $\mathbf{V}^{\mathbf{F}}$ without specifying an evolutionary dynamic $\mathbf{V}$.

From comparison between this and GH-ii), and we find that the second factor does not appear in the derivative $\partial G/\partial \pi$ of the aggregate *net* gain. Net gains for those marginal agents are negligible, i.e., they are zero before the change $\Delta\pi$ occurs. Thus, the changes in their switching decisions have only negligible impact on G.

In a nutshell, non-monotonic change of $\Gamma^{\mathbf{F}}$ stems from failure of the *gross* gain to be an exact determinant for an agent's switching decision. Inertia and suboptimal switches prevent agents from fully exploiting the possible gross gain. These distortions are, however, taken into account in the *net* gain as costs and constraints. Evolutionary agents *fully* exploit *net* gains; the net gain explains agents' switching decision completely under economically evolutionary dynamics. We could say net gains capture the *true* "greed" or incentives behind switching decisions as the essential driving force of economically reasonable evolutionary dynamics.

6.2 Why not assuming payoff monotonicity?

Our notion of economic reasonability requires strategy revision to be justifiable from the optimization principle by introducing additional costs and constraint on available actions. This notion is applicable beyond the BRD thanks to flexibility of our framework. The preceding literature on evolutionary dynamics attempted to capture consistency of evolutionary dynamics with incentives by imposing *payoff monotonicity*, i.e., positive relationship between the change in the share of each action and the relative payoff of the action and then found counterexamples on the link between static and dynamic stability. Friedman (1991) defines *order compatibility* of a game dynamic $\mathbf{V}^{\mathbf{F}}$ as a property such as i) $\dot{x}_a > \dot{x}_b \Leftrightarrow F_a(\mathbf{x}) > F_b(\mathbf{x})$ whenever $x_a, x_b > 0$ and ii) $\dot{x}_u = 0$ whenever $x_u = 0$. In his Counterexample 2, Friedman suggests that some order compatible dynamics diverge away from a unique interior ESS in a linear game, which is indeed a strictly contractive game.[41]

The comparison between his general study and ours clarifies the source of this difference in in-/stability of an ESS. We share the desiderata that economic dynamics should agree with incentives. In particular, if all the actions are always available, i.e., $\mathbb{P}_{\mathcal{A}a}(\mathcal{A} \setminus \{a\}) \equiv 1$ and Q is strictly increasing, then an agent who is taking a worse action is more likely to abandon it than one taking a better action. So we can

[41] One may replace condition i) with monotonicity of percentage growth rate (Sandholm, 2010b, §5.4.5): $\dot{x}_a/x_a > \dot{x}_b/x_b \Leftrightarrow F_a(\mathbf{x}) > F_b(\mathbf{x})$. The replicator dynamic satisfies this condition but fails to guarantee convergence to an equilibrium under (null) contractive games, as we argue in Section 6.3.

say our notion of economic reasonability implies some sort of payoff monotonicity at individual level. In contrast, the order compatibility imposes payoff monotonicity on the aggregate net flow $\dot{\mathbf{x}}$ in the entire population. Because of this difference, an order compatible dynamic may not fit into our framework. It is indeed the case for the dynamics that negate convergence to a regular ESS in Friedman's example. On the other hand, since the available action set may be restricted in our framework, our dynamic may fail order compatibility. See Supplement S5.2.

6.3 Why not covering imitative dynamics?

Assumption A0 prevents us from including observational dynamics into our framework: for example, we could define the probability that action b is available, i.e., $\mathbb{P}_{\mathcal{A}a}(\{\mathcal{A}'_a \mid b \in \mathcal{A}'_a\})$ for imitative dynamic as the probability that a b-player is sampled from the society and then we could define the individual gain function g as in (6) and then the aggregate G. But the essence of such observational dynamics is that the sampling probability depends on the current actual distribution of actions in the society. Then, Assumption A0 does not hold, which invalidates property GH.

For example, consider the replicator dynamic; it could fit into our framework by having $\mathbb{P}_{\mathcal{A}a}(\{b\}) = x_b$. If we dare to apply our definition of the aggregate net gain, we obtain

$$G(\mathbf{x})[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a \sum_{b\in\mathcal{A}} x_b \mathbb{E}_Q[\pi_b - \pi_a - q]_+.$$

We can readily find that the time derivative of this aggregate gain function is

$$\dot{G}(\mathbf{x})[\boldsymbol{\pi}] = \dot{\boldsymbol{\pi}} \cdot \underbrace{\dot{\mathbf{x}}}_{\partial G/\partial \boldsymbol{\pi}} + \underbrace{H(\mathbf{x})[\boldsymbol{\pi}] + \sum_{a\in\mathcal{A}} x_a \sum_{b\in\mathcal{A}} \dot{x}_b \mathbb{E}_Q[\pi_b - \pi_a - q]_+}_{\dot{\mathbf{x}}\cdot\partial G/\partial \mathbf{x}}.$$

The last term represents the effects of changes in $\mathbf{x}$ through changes in the sampling probability $\mathbb{P}_{\mathcal{A}a}(\{b\}) = x_b$ and thus appears peculiarly for observational dynamics.[42] Due to this term, $\dot{G}$ may not be negative and G may not monotonically decrease over time even in a contractive game; see the example in Supplement S5.3.

For major imitative dynamics such as the replicator and projection dynamics,

[42]Notice that $\dot{x}_b$ may be positive or negative, depending on the share of b players (and thus the likelihood of having them sampled) and the relative payoff of b. If b performs better than other strategies and also has got a large share of players, this additional term should be positive while the other terms are negative.

only Lyapunov stability is obtained for contractive games in general; see the remark in Supplement S5.3. Lahkar and Sandholm (2008); Sandholm et al. (2008); Mertikopoulos and Sandholm (2018) explore this issue. For asymptotic stability, *strict* contraction is needed. Requirement of strictness of contractive games may appear to be a subtle, minor difference. However, we should notice that it excludes the simplest situation that we expect to assure equilibrium stability, i.e., a game with a constant payoff vector $\mathbf{F}(\mathbf{x}) \equiv \boldsymbol{\pi}^0$ at any $\mathbf{x} \in \mathcal{X}$. We naturally expect economic agents to switch to the best actions, which do not change over time in this situation. Agents should take those best actions in the long run: that is, the population should converge to a Nash equilibrium. However, it is known that imitative dynamics result in a cyclic trajectory, keeping away from the equilibrium.[43] The driving force behind imitative dynamics cannot be understood only by incentives.

7 Concluding remarks

In this paper, we present a unifying and economically intuitive approach to link static and dynamic stability based on economic reasonability behind agents' updates of choices. The approach is summarized as follows:

1. Reconstruct an individual agent's decision rule (revision protocol) based on optimization; distortions from exact optimization can be justified by introducing additional costs and constraints.

2. Then, calculate an agent's expected *net* gain as the maximal payoff improvement after deducting those additional costs given the additional constraints. Aggregate it over all the agents.

3. When static stability is granted, the aggregate net gain should decrease to zero and thus it should work as a Lyapunov function to prove dynamic stability.

Our general approach not only provides a general proof of dynamic stability with mathematically universal and economically intuitive logic but also extends it to the heterogeneous setting where different agents follow different decision rules or have different payoff functions.

[43] This is generally proven for null contractive games (including zero-sum games), where Equation (4) always holds with equality; see Sandholm (2010b, Sec. 9.1).

Our notion of economic reasonability is new in evolutionary game theory, but prevalent in any other areas of economics. The optimization principle of "rational choice" is so powerful to reveal the underlying preference, which in turn allows us to empirically test rationality (Kitamura and Stoye, 2018). In estimation of discrete choice models, econometricians attempt to explain empirical data with random utility models (Anderson et al., 1992). In game experiments, quantal response equilibrium has been dominant in solution concepts (McKelvey and Palfrey, 1995). In recent development of decision theory, theorists try to rationally axiomatize known behavioral distortions.[44] Those economists admit distortions from exact optimization but try to understand each of those distortions by revealing hidden costs, constraints and structures, and have proven usefulness of the approach by deriving rich implications and presenting good matches with empirical data. Our approach shares this spirit and proves usefulness in dynamic modeling by solving a classical and fundamental problem of finding a link between static and dynamic stability.

Looking back at past, one might recall tâtonnement processes in general equilibrium analysis as an ancestor evolutionary dynamics in games. It seems that, despite massive accumulation of studies on tâtonnement especially in 1950–60s (Arrow and Hurwicz, 1958), they did not find a single universal and intuitive principle behind a variety of tâtonnement processes.[45] A Lyapunov function was used to prove dynamic stability (Uzawa, 1960; Negishi, 1962) but its economic meaning did not call economists' interests. [46] Our study suggests that it is because the tâtonnement formalizes a price adjustment process in a market; modern economists indeed admit a variety of market structures and would (at least for majority of us) disagree with a single axiomatization. In contrast, majority of economists agree with optimization-based axiomatization of individual decision making at least as a basic principle and (and adopt it as a dominant teaching principle from freshman to graduate). This

[44]Interestingly, Natenzon (2019) reveals complicated rational choice (Bayesian update of belief in true payoffs from observed payoffs as signals) in frogs' mating choices, which biologists regard as an evidence of irrationality (as they did not think of Bayesian learning) and a support for natural selection.

[45]Mas-Colell et al. (1995, p.621) states "one of the difficulties in this area [tâtonnement stability] is the plethora of plausible disequilibrium models. Although there is a single way to be in equilibrium, there are many different ways to be in disequilibrium." See Kirman (2006, p.262) for a critical survey of the literature on tâtonnement stability.

[46]Weintraub (1991, Ch.4) is a retrospect of historical development of usage of Lyapunov functions in economic theory; Weintraub rediscovered Yasui (1950) as the very first study to apply Lyapunov's dynamic stability theory to the analysis of market equilibrium.

difference in what to focus on and formalize in the dynamic process divides our modeling strategy and tâtonnement's. We have found greed and ex-ante net gains as a universal force and measure behind economically reasonable dynamics just like "gravity" and "height" in kinetics. Is there any such thing for a market?[47]

At present, one of emerging applications of evolutionary dynamics is an analytical foundation of agent-based simulations and distributional controls.[48] Such computational models run on a discrete time horizon and involve more or less of random perturbations. We can connect our results smoothly to discrete-time dynamics. See Roth and Sandholm (2013) for legitimacy of using continuous-time large-population dynamics based on optimization for approximation of discrete dynamics, and Ellison et al. (2016) for verification of bounded convergence time by a Lyapunov function. Further, the construction of evolutionary dynamics in this paper does not require an agent to have ad hoc knowledge of the payoff structure or even the current payoffs of all feasible choices or to observe other agents' choices. This parsimony in agents' information may be useful for design of distributional controls over decentralized systems with a large number of feasible choices, such as implementation of optimal network formation and traffic flows.

While our analysis is confined here to a normal-form game, the author believes that there is a broad range of future applications of our approach to proving equilibrium stability beyond this basic setting; see Sawa and Zusai (2019) for an example of our approach to the BRD with a nested choice structure in multitasking environments. Typically in sequential-move, repeated and coalitional games, one may define a version of static stability that is tailored to each specific situation in order to refine the equilibrium concept.[49] The equilibrium refinement with static stability may be based on a rough idea on how an agent learns plays of others in the

[47]It suggests that, even for a classical market model, our approach may be applicable to derive dynamic stability as long as the dynamic formalizes an individual decision making. Toda (2015) considers a market equilibrium model with (possibly) myopic non-optimizing behavioral agents. While his paper focuses on the existence of a stationary rational expectation equilibrium and does not formally define a dynamic, the underlying idea behind the stationarity condition can be regarded as being built upon aggregation of myopic demand decisions. So, our approach may be applicable to his model.

[48]See Izquierdo et al. (2018, 2019) for evolutionary game theorists' applications to agent-based simulations and Marden and Shamma (2015) for applications to distributional controls.

[49]See Garcia and van Veelen (2016); Fujiwara-Greve and Okuno-Fujiwara (2009, 2016) for examples in repeated games. Models of preference evolution also normally take the "indirect evolution approach": see Dekel et al. (2007); Alger and Weibull (2013). See Demuynck et al. (2019) for an example of a general definition of static coalitional stability based on myopic adjustments.

society and adjusts its own behavior to it. One may hope to justify the ad-hoc refinement by establishing dynamic stability. This paper suggests to formally define such a dynamic from that rough idea about the underlying adjustment process and track down the changes in the remaining *net gain* under the dynamic; then dynamic stability is verified.

Here we present our approach in the context of evolutionary dynamics, which typically assume myopic decision making.[50] To keep it economically natural, we assume Assumption A1-ii) and obtain property g1, which implies that agents steadily exploit the remaining room for arbitrage, i.e., the ex-ante net gain as much as possible (given the draw of an available strategy set) by *myopically* choosing the best available action based on the current payoff vector. This has been enough and useful for our theorem, but one may hope to enrich evolutionary dynamics by incorporating some sorts of expectation. For this, we may consider an economically natural decision rule (revision process) in case that Assumption A1-ii) is not met. Actually, expectation plays a crucial role in many of interesting economic problems such as Bayesian learning under incomplete information and intertemporal choices as in macroeconmic models. Or alternatively, the separatist's view suggests that we may keep using the conventional modeling of evolutionary dynamics by regarding $\boldsymbol{\pi}_t$ as expected present values or some predictors of future payoffs that are updated based on the current payoffs, as we argued at the end of Section 4.[51] As suggested by Fox and Shamma (2013), we may further separate the belief update process from an evolutionary dynamic and a game; then, we may find a property in this process that contributes to dynamic stability when combined with static stability of the game and economic reasonability of the evolutionary dynamic. Ambitiously we can foresee a future development of our approach in extension of evolutionary dynamics by incorporating more complex and forward-looking structures into economic reasonable dynamics. Yet, the notion of net gains assures dynamic stability in an economically intuitive way.

[50] Perfect foresight dynamics Matsui and Matsuyama (1995) and rationalizable foresight dynamics Matsui and Oyama (2006) are exceptions, though they would be $\mathbf{V}^{\mathbf{F}}$ in our sense and thus need separation for clearer and more general analysis, as we argue below.

[51] Leslie et al. (2018) consider a best response dynamic in a stochastic zero-sum game, where agents take the best response based on such a predictor that is adjusted in response to the current payoff, and prove equilibrium stability.

A1 Appendix to Section 4

Proof of Lemma 1. **g0.** i) is immediate from the definition of g_{a*}. Note that the term $\mathbb{E}_Q[\pi_*(\mathcal{A}'_a) - \pi_a - q]_+$ reflects the fact that a revising agent switches to a different new action only if $\pi_*(\mathcal{A}'_a) \geq \pi_a$; so does this non-negativity of g_{a*}.

For the "only-if" part of ii), first assume $a \in b_*[\boldsymbol{\pi}]$. Then, $\pi_a \geq \pi_*(\mathcal{A}'_a) \geq \pi_*(\mathcal{A}'_a) - q$ for any $\mathcal{A}'_a \subset \mathcal{A} \setminus \{a\}$, as long as $q \geq 0$. This implies $[\pi_*(\mathcal{A}'_a) - \pi_a - q]_+ = 0$ almost surely by Assumption Q1-i). Thus, $g_{a*}[\boldsymbol{\pi}] = 0$.

For the "if" part of ii), we prove its contrapositive by assuming the existence of $b \in \mathcal{A}$ such that $\pi_b > \pi_a$ and then deriving $g_{a*}[\boldsymbol{\pi}] > 0$. Assumption A1-i) assures the existence of $\mathcal{A}'_a \subset \mathcal{A} \setminus \{a\}$ such that $\mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) > 0$ and $b \in \mathcal{A}'_a$. For this set $\mathcal{A}'_a$, we have $\pi_*(\mathcal{A}'_a) \geq \pi_b > \pi_a$. Then, by Assumption Q1, it is guaranteed that $Q(\pi_*(\mathcal{A}'_a) - \pi_a) > 0$ and thus $\mathbb{E}_Q[\pi_*(\mathcal{A}'_a) - \pi_a - q]_+ > 0$. Hence, we have $g_{a*}[\boldsymbol{\pi}] > 0$.

g1. Make a partition of $\mathcal{A}$ according to $\boldsymbol{\pi}$, say $\mathcal{A}_1, \mathcal{A}_2, \ldots$ such as

$$[a, a' \in \mathcal{A}_i \Leftrightarrow \pi_a = \pi_{a'}] \quad \text{and} \quad [a \in \mathcal{A}_i \text{ and } a' \in \mathcal{A}_{i'} \text{ with } i < i' \Leftrightarrow \pi_a > \pi_{a'}].$$

Let π_i be the payoff obtained from actions in set $\mathcal{A}_i$: i.e., π_i is defined as π_a with some (and indeed all) $a \in \mathcal{A}_i$. $\mathcal{A}_i$ is the set of actions that yield the i-th greatest payoff π_i among $\{\pi_a \mid a \in \mathcal{A}\}$.

Define $p^i_{\mathcal{A}a}$ by

$$p^i_{\mathcal{A}a} := \mathbb{P}_{\mathcal{A}a}(\{\mathcal{A}'_a \subset \mathcal{A} \setminus \{a\} \mid \mathcal{A}'_a \cap \mathcal{A}_i \neq \emptyset \text{ and } \mathcal{A}'_a \cap \mathcal{A}_j = \emptyset \text{ for any } j < i\}) \qquad \text{for each } i.$$

Given current action a, the maximal available payoff after the revision becomes π_i with probability $p^i_{\mathcal{A}a}$. For the set $\mathcal{A}'_a$ in the above definition of $p^i_{\mathcal{A}a}$, we have $\pi_*(\mathcal{A}'_a) = \pi_i$. Assumption A1-i) implies $p^1_{\mathcal{A}a} > 0$.

Denote by $I(a)$ the index of the partition that a belongs to: i.e., $a \in \mathcal{A}_{I(a)}$. Notice that

$$g_{a*}(\boldsymbol{\pi}) = \sum_{i=1}^{I(a)-1} p^i_{\mathcal{A}a} \mathbb{E}_Q[\pi_i - \pi_a - q]_+.$$

If $\pi_a \leq \pi_b$, then $I(a) \geq I(b)$ as well as

$$\mathbb{E}_Q[\pi_i - \pi_a - q]_+ \geq \mathbb{E}_Q[\pi_i - \pi_b - q]_+ \text{ for any } i. \tag{9}$$

Fix $I < I(b)$ arbitrarily. Since $a, b \notin \bigcup_{i=1}^{I} \mathcal{A}_i$ as $I < I(b) \leq I(a)$, we obtain

$$\sum_{i=1}^{I} p^i_{\mathcal{A}a} = \mathbb{P}_{\mathcal{A}a}\left(\{\mathcal{A}'_a \mid \mathcal{A}'_a \cap (\bigcup_{i=1}^{I} \mathcal{A}_i) \neq \emptyset\}\right) = \mathbb{P}_{\mathcal{A}b}\left(\{\mathcal{A}'_b \mid \mathcal{A}'_b \cap (\bigcup_{i=1}^{I} \mathcal{A}_i) \neq \emptyset\}\right) = \sum_{i=1}^{I} p^i_{\mathcal{A}b}$$

for all $I < I(b)$ by applying Assumption A1-ii) to $\mathcal{A}_{ab} = \bigcup_{i=1}^{I} \mathcal{A}_i$. Hence, we have

$$p^i_{\mathcal{A}a} = p^i_{\mathcal{A}b} \quad \text{for any } i < I(b). \tag{10}$$

Combining this with (9), we obtain part i):

$$\begin{aligned} g_{a*}[\boldsymbol{\pi}] &= \sum_{i=1}^{I(b)-1} p^i_{\mathcal{A}a} \mathbb{E}_Q[\pi_i - \pi_a - q]_+ + \sum_{i=I(b)}^{I(a)-1} p^i_{\mathcal{A}a} \mathbb{E}_Q[\pi_i - \pi_a - q]_+ \\ \geq g_{b*}[\boldsymbol{\pi}] &= \sum_{i=1}^{I(b)-1} p^i_{\mathcal{A}b} \mathbb{E}_Q[\pi_i - \pi_b - q]_+. \end{aligned}$$

Further, if $\pi_a < \pi_b$, then $I(a) > I(b)$. As $\pi_a < \pi_b \leq \pi_1$, Assumption Q1 implies $Q(\pi_1 - \pi_a) > 0$ and thus $\mathbb{E}_Q[\pi_1 - \pi_a - q]_+ > \mathbb{E}_Q[\pi_1 - \pi_b - q]_+$. With the fact $p^1_{\mathcal{A}a} > 0$, it guarantees $g_{a*}[\boldsymbol{\pi}] > g_{b*}[\boldsymbol{\pi}]$, i.e., part ii) of property g1.

g2. First of all, let $\mathfrak{b}(\mathcal{A}'_a) \in \mathcal{A}'_a$ be a choice of a (not necessarily optimal) action for non-empty available action set $\mathcal{A}'_a \subset \mathcal{A} \setminus \{a\}$; we could call $\mathfrak{b} : 2^{\mathcal{A}\setminus\{a\}} \setminus \emptyset \to \mathcal{A} \setminus \{a\}$ a policy function for a revising agent to make a choice after observing (non-empty) available action set. Let $\mathfrak{B}$ be the set of all such policy functions; notice that it is a finite set as long as $\mathcal{A}$ is a finite set. Define the ex-ante gain from policy $\mathfrak{b}$, $g_{a\mathfrak{b}}$, by

$$g_{a\mathfrak{b}}[\boldsymbol{\pi}] := \sum_{\mathcal{A}'_a \subset \mathcal{A}\setminus\{a\}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) \mathbb{E}_Q[\pi_{\mathfrak{b}(\mathcal{A}'_a)} - \pi_a - q]_+.$$

If an agent follows the randomly constrained optimization protocol (2), $\mathfrak{b}(\mathcal{A}'_a)$ should be chosen from $b_*(\mathcal{A}'_a)$ in each possible realization of $\mathcal{A}'_a$; thus,

$$g_{a*}[\boldsymbol{\pi}] = \max_{\mathfrak{b} \in \mathfrak{B}} g_{a\mathfrak{b}}[\boldsymbol{\pi}].$$

Specifically, the maximum is attained by a policy $\mathfrak{b}_* \in \mathfrak{B}$ such that $\mathfrak{b}_*(\mathcal{A}'_a) \in b_*(\mathcal{A}'_a) \subset \mathcal{A}'_a$.

With $\mathfrak{b}$ fixed arbitrarily in $\mathfrak{B}$, function $g_{a\mathfrak{b}}$ is Lipschitz continuous and thus differ-

entiable almost everywhere. It is differentiable at $\boldsymbol{\pi}$ if and only if $\mathbb{P}_Q$ is continuous at $\pi_{\mathfrak{b}(\mathcal{A}'_a)} - \pi_a$ of all $a \in \mathcal{A}$ and all $\mathcal{A}'_a \subset \mathcal{A} \setminus \{a\}$ with $\mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) > 0$. Then, the derivative is

$$\frac{\partial g_{a\mathfrak{b}}}{\partial \pi_b}[\boldsymbol{\pi}] = \begin{cases} -\sum_{\mathcal{A}'_a \subset \mathcal{A}\setminus\{a\}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) Q(\pi_{\mathfrak{b}(\mathcal{A}'_a)} - \pi_a) \in [-1, 0] & \text{if } b = a, \\ \sum_{\substack{\mathcal{A}'_a \subset \mathcal{A}\setminus\{a\} \\ \text{s.t. } b \in \mathfrak{b}(\mathcal{A}'_a)}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) Q(\pi_{\mathfrak{b}(\mathcal{A}'_a)} - \pi_a) \in [0, 1] & \text{otherwise} \end{cases}$$

for each action $b \in \mathcal{A}$. Therefore,

$$\frac{\partial g_{a\mathfrak{b}}}{\partial \boldsymbol{\pi}}[\boldsymbol{\pi}]\Delta\boldsymbol{\pi} = \sum_{\mathcal{A}'_a \subset \mathcal{A}\setminus\{a\}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) Q(\pi_{\mathfrak{b}(\mathcal{A}'_a)} - \pi_a)(\Delta\pi_{\mathfrak{b}(\mathcal{A}'_a)} - \Delta\pi_a) \quad \text{for any } \Delta\boldsymbol{\pi} \in \mathbb{R}^A.$$

As $g_{a\mathfrak{b}}$ is Lipschitz continuous on $\mathbb{R}^A$, the maximal value function g_{a*} is also Lipschitz continuous and thus differentiable almost everywhere in $\mathbb{R}^A$ by a version of Danskin's envelop theorem in Hofbauer and Sandholm (2009, Theorem A.4). By applying the above equation to $\mathfrak{b} = \mathfrak{b}_*$ and noticing $\pi_{\mathfrak{b}_*(\mathcal{A}'_a)} = \pi_*(\mathcal{A}'_a)$, we obtain

$$\frac{dg_{a*}}{d\boldsymbol{\pi}}[\boldsymbol{\pi}]\Delta\boldsymbol{\pi} = \sum_{\mathcal{A}'_a \subset \mathcal{A}\setminus\{a\}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) Q(\pi_*(\mathcal{A}'_a) - \pi_a)(\Delta\pi_*(\mathcal{A}'_a) - \Delta\pi_a) \quad \text{for any } \Delta\boldsymbol{\pi} \in \mathbb{R}^A.$$

Note that g_{a*} is differentiable at $\boldsymbol{\pi}$ if and only if $\mathbb{P}_Q$ is continuous at $\pi_*(\mathcal{A}'_a) - \pi_a$ for all $a \in \mathcal{A}$ and all $\mathcal{A}'_a \subset \mathcal{A} \setminus \{a\}$ with $\mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) > 0$. In this case, any $\mathbf{z}_a \in \mathbf{V}_a(\mathbf{x})[\boldsymbol{\pi}]$ satisfies

$$\mathbf{z}_a \cdot \Delta\boldsymbol{\pi} = \sum_{\mathcal{A}'_a \subset \mathcal{A}\setminus\{a\}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) Q(\pi_*(\mathcal{A}'_a) - \pi_a)(\mathbf{y}_a(\mathcal{A}'_a) - \mathbf{e}_a) \cdot \Delta\boldsymbol{\pi}.$$

with some $\mathbf{y}_a(\mathcal{A}'_a) \in \Delta^{\mathcal{A}}(b_*[\boldsymbol{\pi}](\mathcal{A}'_a))$. We have $\mathbf{y}_a(\mathcal{A}'_a) \cdot \Delta\boldsymbol{\pi} = \Delta\pi_*(\mathcal{A}'_a)$; and, $\mathbf{e}_a \cdot \Delta\boldsymbol{\pi} = \Delta\pi_a$. Combining these, we obtain (7). □

Proof of Lemma 2. **h.** Part i) is immediate from property g1-i). For part ii), first consider action a such that $a \in b_*[\boldsymbol{\pi}]$. Then, any other action $b \in \mathcal{A}$ yields $\pi_b \leq \pi_a$; by property g1-i) again, it is equivalent to $g_{b*}[\boldsymbol{\pi}] \geq g_{a*}[\boldsymbol{\pi}]$; thus, $g_{**}[\boldsymbol{\pi}](\mathcal{A}'_a) = \min_{b \in \mathcal{A}'_a} g_{b*}[\boldsymbol{\pi}] \geq g_{a*}[\boldsymbol{\pi}]$. By (8), it implies $h_{a*}[\boldsymbol{\pi}]$ cannot be strictly negative. By part i) of this property h, it follows that $h_{a*}[\boldsymbol{\pi}] = 0$. For the opposite, assume $h_{a*}[\boldsymbol{\pi}] < 0$. Again, we can see from (8) that it implies the existence of $\mathcal{A}'_a$ such that $\mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) > 0, Q(\pi_*(\mathcal{A}'_a) - \pi_a) > 0$ and $g_{**}[\boldsymbol{\pi}](\mathcal{A}'_a) - g_{a*}[\boldsymbol{\pi}] < 0$. By property g1-

ii), the last condition implies $\pi_*(\mathcal{A}'_a) > \pi_a$; there exists an action in $\mathcal{A}'_a \subset \mathcal{A} \setminus \{a\}$ that is better than a. Therefore, $a \notin b_*[\boldsymbol{\pi}]$. Finally, since Q is upper semicontinuous and $\pi_*(\mathcal{A}'_a) - \pi_a$ is a continuous function of $\boldsymbol{\pi}$, the composite $Q(\pi_*(\mathcal{A}'_a) - \pi_a)$ is upper semicontinuous in $\boldsymbol{\pi}$. Since $g_{**}[\boldsymbol{\pi}](\mathcal{A}'_a) - g_{a*}[\boldsymbol{\pi}]$ is continuous and nonpositive, this implies lower semicontinuity of h_{a*}.

gh. First of all, notice that differentiablity of g_{a*} at $\boldsymbol{\pi}$ implies continuity of $\mathbb{P}_Q$ at $\pi_*(\mathcal{A}'_a) - \pi_a$ for all $a \in \mathcal{A}$ and all $\mathcal{A}'_a \subset \mathcal{A} \setminus \{a\}$ with $\mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) > 0$. Any actions in $b_*[\boldsymbol{\pi}](\mathcal{A}'_a)$ yield the greatest payoff $\pi_*(\mathcal{A}'_a)$ among all the available actions in $\mathcal{A}'_a$; property g1 implies that they yield the smallest first-order gain $g_{**}[\boldsymbol{\pi}](\mathcal{A}'_a)$ among them. Thus, any mixture of those actions $\mathbf{y}_a(\mathcal{A}'_a) \in \Delta^{\mathcal{A}}(b_*[\boldsymbol{\pi}](\mathcal{A}'_a))$ satisfies

$$\mathbf{y}_a(\mathcal{A}'_a) \cdot \mathbf{g}_*[\boldsymbol{\pi}] = g_{**}[\boldsymbol{\pi}](\mathcal{A}'_a).$$

By the fact $\mathbf{e}_a \cdot \mathbf{g}_*[\boldsymbol{\pi}] = g_{a*}[\boldsymbol{\pi}]$ and the assumption $\mathbf{z}_a \in \mathbf{V}_a[\boldsymbol{\pi}]$, we find from (8) that this implies the equation in property gh. □

Proof of Theorem 2. **G.** i) is immediately obtained from property g0-i) and the fact $\mathbf{x} \in \mathbb{R}^A_+$. For ii), notice that $\mathbf{x} \in \Delta^{\mathcal{A}}(b_*[\boldsymbol{\pi}])$ is equivalent to $x_a > 0 \Rightarrow a \in b_*[\boldsymbol{\pi}]$; and, by property g0-i), $G(\mathbf{x})[\boldsymbol{\pi}] = 0$ is equivalent to $[x_a > 0 \Rightarrow g_{a*}[\boldsymbol{\pi}] = 0]$. The equivalence between $a \in b_*[\boldsymbol{\pi}]$ and $g_{a*}[\boldsymbol{\pi}] = 0$ has been established in property g0-ii). Thus, combining these, we obtain property G-ii).

H. Each of the two parts can be readily verified from property h, similarly to the proof of property G.

GH. Suppose that G is differentiable at $(\mathbf{x}, \boldsymbol{\pi})$. i) Since $\Delta\mathbf{x} \in \mathbf{V}(\mathbf{x})[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a \mathbf{V}_a[\boldsymbol{\pi}]$ by (3), there exists $(\mathbf{z}_a)_{a\in\mathcal{A}}$ such that $\Delta\mathbf{x} = \sum_{a\in\mathcal{A}} x_a \mathbf{z}_a$ and $\mathbf{z}_a \in \mathbf{V}_a[\boldsymbol{\pi}]$ for each a. By property gh, we have

$$\Delta\mathbf{x} \cdot \mathbf{g}_*[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a \mathbf{z}_a \cdot \mathbf{g}_*[\boldsymbol{\pi}] = \sum_{a\in\mathcal{A}} x_a h_{a*}[\boldsymbol{\pi}] = H(\mathbf{x})[\boldsymbol{\pi}].$$

By Assumption A0, $\mathbf{g}_*[\boldsymbol{\pi}]$ is constant to $\mathbf{x}$; thus, $G(\mathbf{x})[\boldsymbol{\pi}] = \mathbf{g}_*[\boldsymbol{\pi}] \cdot \mathbf{x}$ is linear in $\mathbf{x}$ with coefficient $\mathbf{g}_*[\boldsymbol{\pi}]$, when $\boldsymbol{\pi}$ is fixed. Thus, $\partial G/\partial \mathbf{x} \equiv \mathbf{g}_*[\boldsymbol{\pi}]$. Plugging this into the above equation, we obtain part i) of property GH.

ii) As $G(\mathbf{x})[\boldsymbol{\pi}] = \mathbf{x} \cdot \mathbf{g}_*[\boldsymbol{\pi}]$, property g2 implies

$$\frac{\partial G}{\partial \boldsymbol{\pi}}(\mathbf{x})[\boldsymbol{\pi}]\Delta\boldsymbol{\pi} = \sum_{a\in\mathcal{A}} x_a \frac{\partial \mathbf{g}_{a*}}{\partial \boldsymbol{\pi}}[\boldsymbol{\pi}]\Delta\boldsymbol{\pi} = \sum_{a\in\mathcal{A}} x_a \mathbf{z}_a \cdot \Delta\boldsymbol{\pi}$$

for any $\mathbf{z}_a \in \mathbf{V}_a[\boldsymbol{\pi}]$. Since $\Delta\mathbf{x} \in \mathbf{V}[\boldsymbol{\pi}](\mathbf{x}) = \sum_{a\in\mathcal{A}} x_a \mathbf{V}_a[\boldsymbol{\pi}]$ by (3), this implies part ii) of property GH. □

Proof of Theorem 3. We use function G as the Lyapunov function W in Theorem 1 and function H as the decay rate function $\tilde{W}$. Thanks to continuous differentiability (and thus Lipschitz continuity) of $\mathbf{F}$, Lipschitz continuity of G and lower semicontinuity of H are succeeded to those of $G^{\mathbf{F}}$ and $H^{\mathbf{F}}$, respectively. Properties G-i and H-i guarantee condition (a) in the theorem. By properties G-ii and H-ii, condition (b) is satisfied with the limit set being $X^* = \mathrm{NE}(\mathbf{F})$.

By the chain rule and then property GH, we have

$$\begin{aligned}\frac{d}{dt}G^{\mathbf{F}}(\mathbf{x}_t) &\equiv \frac{\partial G}{\partial \mathbf{x}}(\mathbf{x}_t)[\mathbf{F}(\mathbf{x}_t)]\dot{\mathbf{x}}_t + \frac{\partial G}{\partial \boldsymbol{\pi}}(\mathbf{x}_t)[\mathbf{F}(\mathbf{x}_t)]\frac{d\mathbf{F}}{d\mathbf{x}}(\mathbf{x}_t)\dot{\mathbf{x}}_t \\ &= \dot{\mathbf{x}}_t \cdot \frac{d\mathbf{F}}{d\mathbf{x}}(\mathbf{x}_t)\dot{\mathbf{x}}_t + H^{\mathbf{F}}(\mathbf{x}_t) \qquad (11)\end{aligned}$$

for any $\dot{\mathbf{x}}_t \in \mathbf{V}^{\mathbf{F}}(\mathbf{x}_t)$ and $\mathbf{x}_t \in \mathcal{X}$. If $D\mathbf{F}$ is negative semidefinite at $\mathbf{x}_t$, (11) implies

$$\frac{d}{dt}G^{\mathbf{F}}(\mathbf{x}_t) \le H^{\mathbf{F}}(\mathbf{x}_t) \le 0 \qquad \text{for any } \dot{\mathbf{x}}_t \in \mathbf{V}^{\mathbf{F}}(\mathbf{x}).$$

Thus we obtain (5) without making restriction on the initial state $\mathbf{x}_0$; so it holds globally and thus the basin of attraction is $\mathcal{X}$. This proves part i). By the same token, we can verify the local stability as in part ii-a) by setting X' to a neighborhood of $\mathbf{x}^*$ where $\mathbf{x}^*$ is the only Nash equilibrium and $D\mathbf{F}$ is negative semidefinite.

For part ii-b), consider a Carathéodory solution path $\{\mathbf{x}_t\}_{t\in\mathbb{R}_+}$ from $\mathbf{x}_0 = \mathbf{x}^*$. Then, we have $\dot{G}(\mathbf{x}_0) = H(\mathbf{x}^*) + \dot{\mathbf{x}}_0 \cdot D\mathbf{F}(\mathbf{x}^*)\dot{\mathbf{x}}_0$. The former term $H(\mathbf{x}^*)$ is zero by property H-ii), since $\mathbf{x}^*$ is a Nash equilibrium. The assumption for part ii-b) allows us to take $\dot{\mathbf{x}}_0 \in \mathbf{V}^{\mathbf{F}}(\mathbf{x}^*)$ that makes the latter term $\dot{\mathbf{x}}_0 \cdot D\mathbf{F}(\mathbf{x}^*)\dot{\mathbf{x}}_0$ being positive. With $G(\mathbf{x}_0) \ge 0$ (indeed zero by property G-ii)), this implies $G(\mathbf{x}_t) > 0$ for any t in some interval of time $(0, T]$. Property G-ii) implies that $\mathbf{x}_t$ in this interval of time cannot be at a Nash equilibrium. Thus, it must be escaping from $\mathbf{x}^*$; this implies that $\mathbf{x}^*$ is not Lyapunov stable. □

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Supplement to "Gains in evolutionary dynamics"

Dai Zusai

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S1 Supplement to Section 2

S1.1 Supplement to Section 2.2

Examples of economically reasonable dynamics in the basic set-up

In the following examples, $\mathbb{P}_{Aa}(\mathcal{A}'_a)$ depends only on the number of available actions $\#\mathcal{A}'_a$, but not on current action a or what actions are in $\mathcal{A}'_a$. That is, there exists $\{p_n\}_{n=1}^{A-1}$ such that $\mathbb{P}_{Aa}(\mathcal{A}'_a) = p_n$ whenever $\#\mathcal{A}'_a = n$.[1]

Example S1 (Best response dynamics). If $p_{A-1} = 1$, all actions are available. An agent compares payoffs of all actions and then switches to the optimal (best response) action if the agent abandons the current action. If we set $Q(q) = \mathbf{1}\{q \geq 0\}$, the switching rate is (almost) always zero and thus the conditional switching rate is 1 regardless of the amount of the gross gain. The combination of them yields exactly the *standard best response dynamic* (BRD) such as $\dot{\mathbf{x}} \in B(\mathbf{x}) - \mathbf{x}$ (Gilboa and Matsui, 1991; Hofbauer, 1995). By allowing Q to be any cumulative distribution function that satisfies Assumption Q1, it is generalized to a *tempered BRD* (tBRD) (Zusai, 2018). ∎

Example S2 (Pairwise payoff comparison dynamics). If $p_1 = 1/(A-1)$, only one action is available beside the current action. It yields a pairwise payoff comparison dynamic: a revising agent randomly samples an action for a candidate to switch to, compares the current payoff with the payoff from the sampled action and switches to the latter action with some positive probability (conditional switching rate) if and only if the latter yields a higher payoff than the current payoff. Particularly, in *Smith dynamic* (Smith, 1984), the conditional switching rate is proportional to the payoff difference between the randomly sampled action b and the current action a; that is, $Q(q) = c[q]_+$ with some constant $c \in \mathbb{R}_{++}$.[2] We can generalize it so the probability is determined by function $Q : \mathbb{R} \to [0,1]$ that increases in $\mathbb{R}_+$ and satisfies $Q(q) = 0$

[1] To meet $\sum_{\mathcal{A}'_a \subset \mathcal{A}} \mathbb{P}_{\mathcal{A}a}(\mathcal{A}'_a) = 1$, we need $\sum_{n=1}^{A-1} p_n \ {}_{A-1}C_n > 0$.

[2] To interpret $Q(q)$ as a c.d.f., we may want c to be small enough to assure $\sup_{a,b \in \mathcal{A}, \mathbf{x} \in \mathcal{X}} Q(F_a(\mathbf{x}) - F_b(\mathbf{x})) \leq 1$. But, for our theorems, the upper bound on Q does not have to be one and can be any finite positive number.

for any $q < 0$. So, conditional on b randomly picked, the rate of switch from a to b is $Q([\pi_b - \pi_a]_+)$. This dynamic fits with our framework (2) by setting $\mathbb{P}_{\mathcal{A}a}(\{b\}) = 1/(A-1)$ for all $b \in \mathcal{A} \setminus \{a\}$ and defining $\mathbb{P}_Q$ from this switching rate function Q. All the assumptions except Assumption Q1 are guaranteed from this construction. Assumption Q1 is satisfied if $Q(q) > 0$ for all $q > 0$; so does Smith dynamic. ∎

Example S3 (BRD on randomly restricted action set). A straight interpretation of our framework on revision protocols is that an agent chooses an action from a randomly restricted action set as to maximize the net payoff. The assumptions on $\mathbb{P}_{\mathcal{A}\cdot}$ look very general, while the above examples might not fully exploit generality of our framework. So it is worth to explore how we can stretch the reach of our framework beyond these known dynamics with concrete examples of $\mathbb{P}_{\mathcal{A}\cdot}$.

The following examples do not fall neither of the major dynamics mentioned above.

Example S3-i. In a revision opportunity, each action other than the current action becomes available with a certain probability. The availability of each action is independent of those of others and the probability of each action's availability is common to all the actions. This is formulated in $\mathbb{P}_{\mathcal{A}\cdot}$ as $p_n = \bar{p}^n/(1-(1-p)^{A-1})$ with some $\bar{p} \in (0,1)$.[3] The denominator is to exclude the case that no action becomes available.

Example S3-ii. In a revision opportunity, the number of available actions is determined first: n actions will be available with probability $\tilde{p}_n \in (0,1)$ ($1 \leq n \leq A-1$ and $\sum_{n=1}^{A-1} \tilde{p}_n = 1$). Given n the number of available actions, each possible set of n actions is drawn with equal probability as the available action set for the revising agent. This is formulated as $p_n = \tilde{p}_n / {}_{A-1}C_n$.

Interestingly, with the same Q as the standard BRD, these dynamics reduce to an ordinal version of tempered BRDs in which the switching rate depends on the ordinal ranking of a new action based on the current payoff vector. For example, in Example S3-i, an agent switches to the best action with probability $\bar{p}$, to the second best with probability $(1-\bar{p})\bar{p}$, and to the i-th best with probability $(1-\bar{p})^{i-1}\bar{p}$.[4] ∎

[3] Here $\bar{p}^n$ is just $\bar{p}$ raised by power of n.

[4] I would like thank Bill Sandholm to call my attention to this ordinal property.

Assumption A1-ii) allows $\mathbb{P}_{\mathcal{A}\cdot}$ to be asymmetric over different actions—some actions can be more likely to be available than others—as long as the availability does not depend on the current action. This is utilized in the following two examples. Besides, the generalized (and tighter) assumption illustrates the difference from order compatibility, proposed by Friedman (1991); see Section 6.2.

Example S4 (Non-uniform availability). Assume that the action set $\mathcal{A}$ is partitioned to $\mathcal{A}_1, \ldots, \mathcal{A}_I$ (with $I \in \mathbb{N}$).[5] Actions in $\mathcal{A}_i$ is available with probability p_i; let $p_i > 0$ for all i to meet Assumption A1-i). Assume that availability of an action is independent of other actions' availability and that there are at least two actions that are always available; say, $\mathcal{A}_1$ with $p_1 = 1$ and $\#\mathcal{A}_1 \geq 2$.[6] Then, we have

$$\mathbb{P}_{Aa}(\mathcal{A}'_a) = \prod_{i=1}^{I} p_i^{\#(\mathcal{A}_i \cap \mathcal{A}'_a)} \qquad \text{for any } \mathcal{A}'_a \subset \mathcal{A} \setminus \{a\}.$$

It is easy to confirm that this satisfies Assumption A1-ii). ∎

Example S5 (Partitioned available action sets). We could have some actions become available only in a group. Again, consider a partition of $\mathcal{A}$ to $\mathcal{A}_1, \ldots, \mathcal{A}_I$ and assume $\#\mathcal{A}_i \geq 2$ for all i.[7] Either one of those action subsets is randomly picked but not more than one; when $\mathcal{A}_i$ is picked, all the actions in $\mathcal{A}_i$ are available for the agent. Noticing that status-quo action a should be excluded from the available action set (incurring a switching cost), we can formalize this procedure as

$$\mathbb{P}_{Aa}(\mathcal{A}_i \setminus \{a\}) = p_i \qquad \text{with } \sum_{i=1}^{I} p_i = 1.$$

Assume $p_i > 0$ for all i to meet Assumption A1-i); again it is easy to confirm that this satisfies Assumption A1-ii). ∎

[5] As a partition of $\mathcal{A}$, they satisfy $\mathcal{A} = \cup_{i=1}^{I} \mathcal{A}_i$ and $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ if $i \neq j$.

[6] This is to guarantee $\mathbb{P}_{\mathcal{A}a}(\{\mathcal{A}'_a \mid \mathcal{A}'_a \neq \emptyset\}) = 1$ for any a; that is, the available action set $\mathcal{A}'_a$ is non-empty with probability 1 whatever the status-quo action a is.

[7] Without this assumption, we may have $\mathcal{A}_i \setminus \{a\} = \emptyset$. Then, if the status-quo action is a, a revising agent may have no available action (except the status quo) with probability p_i.

S2 Supplement to Section 3

S2.1 Supplement to Section 3.1

Contractive games: global static stability

Definition S1 (Contractive game: Hofbauer and Sandholm, 2009)**.** A population game $\mathbf{F}$ is a **contractive game** if[8]

$$(\mathbf{x}' - \mathbf{x}) \cdot (\mathbf{F}(\mathbf{x}') - \mathbf{F}(\mathbf{x})) \leq 0 \quad \text{for all } \mathbf{x}, \mathbf{x}' \in \mathcal{X}. \tag{S1}$$

If the strict inequality (equality, resp.) holds whenever $\mathbf{x} \neq \mathbf{x}'$, $\mathbf{F}$ is a strict (null, resp.) contractive game.

If $\mathbf{F}$ is C^1, the definition of a contractive game is equivalent to negative semidefiniteness of the Jacobian matrix of $\mathbf{F}$ with respect to the tangent space $T\mathcal{X}$ of the state space $\mathcal{X}$:[9]

$$\Delta\mathbf{x} \cdot D\mathbf{F}(\mathbf{x})\Delta\mathbf{x} \leq 0 \text{ for any } \Delta\mathbf{x} \in T\mathcal{X}, \mathbf{x} \in \mathcal{X}. \tag{S2}$$

So, we would loosely say that $\mathbf{F}$ is a contractive game if and only if $D\mathbf{F}(\mathbf{x})$ satisfies static stability at any $\mathbf{x} \in \mathcal{X}$. That is, a contractive game materializes the notion of global static stability.

Evolutionary stable state: local static stability

In a single-population setting, an **evolutionary stable state** is defined by requiring a Nash equilibrium $\mathbf{x}^*$ to satisfy an additional condition: there is a neighborhood O of $\mathbf{x}^*$ such that

$$(\mathbf{x} - \mathbf{x}^*) \cdot \mathbf{F}(\mathbf{x}) < 0 \qquad \text{for all } \mathbf{x} \in O \setminus \{\mathbf{x}^*\}. \tag{S3}$$

Under Lipschitz continuity of $\mathbf{F}$ (Sandholm, 2010b, Theorem 8.3.5), this definition is equivalent to $\mathbf{x}$ being a Nash equilibrium *and* the existence of a neighborhood O of $\mathbf{x}^*$ such that, for any $\mathbf{x} \in O \setminus \{\mathbf{x}^*\}$,

$$(\mathbf{x} - \mathbf{x}^*) \cdot \mathbf{F}(\mathbf{x}) < 0 \qquad \text{whenever } (\mathbf{x} - \mathbf{x}^*) \cdot \mathbf{F}(\mathbf{x}^*) = 0.$$

[8]This terminology follows Sandholm (2015). Originally, Hofbauer and Sandholm (2009) indeed called it a stable game. It is also called a negative semidefinite game.

[9]For differentiable function $\mathbf{f} = (f_i)_{i=1}^m : \mathbb{R}^n \to \mathbb{R}^m$ with input variable $\mathbf{z} = (z_j)_{j=1}^n$, we denote the Jacobian of $\mathbf{f}$ by $D\mathbf{f}$ or $d\mathbf{f}/d\mathbf{z}$; $\partial f_i/\partial z_j$ locates on the i-th row and the j-th column of $d\mathbf{f}/d\mathbf{z}$.

Consider a linear population game (random matching in a normal form game) with an interior ESS; then, it is a contractive game. If the game is not linear but an ESS lies in the interior $\mathcal{X}$, then the above ESS condition reduces to

$$(\mathbf{x} - \mathbf{x}^*) \cdot (\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}^*)) < 0 \quad \text{for all } \mathbf{x} \in O \setminus \{\mathbf{x}^*\}.$$

since $(\mathbf{x} - \mathbf{x}^*) \cdot \mathbf{F}(\mathbf{x}^*) = 0$ holds for any $\mathbf{x} \in O \setminus \{\mathbf{x}^*\}$ when $\mathbf{x}^*$ is in the interior of $\mathcal{X}$. It suggests that $\mathbf{F}$ behaves like a strictly contractive game around $\mathbf{x}^*$ and thus $D\mathbf{F}(\mathbf{x})$ is negative semidefinite locally at any neighbor state $\mathbf{x}$ around $\mathbf{x}^*$. So, we can say that an interior ESS is a materialization of local (strict) static stability.

A boundary ESS may fail the above condition and thus the negative definiteness of $D\mathbf{F}$ since $(\mathbf{x} - \mathbf{x}^*) \cdot \mathbf{F}(\mathbf{x}^*) = 0$ may not hold for all neighbor $\mathbf{x}$. In the literature on evolutionary dynamics, a *regular* ESS has been investigated as a legitimate extension of an interior ESS to the boundary.

S2.2 Supplement to Section 3.2

Proof of Theorem 1

In a standard Lyapunov stability theorem (e.g. Robinson (1998, §5.5.3), Sandholm (2010b, Corollary 7.B.6)) for a differential equation, a decaying rate function $\tilde{W}$ is not explicitly required while $\dot{W}$ is assumed to be (strictly) negative until $\mathbf{x}$ reaches the limit set X^*. The most significant difference is the requirement of lower semicontinuity of $\tilde{W}$. This assures the existence of a lower bound on the decaying rate $\dot{W}(\mathbf{x}) \leq \bar{w} < 0$ in a hypothetical case in which $\mathbf{x}$ remained out of an arbitrarily small neighborhood of X^* for an arbitrarily long period of time. This excludes the possibility that $\mathbf{x}$ would stay there forever and guarantees convergence to X^* (not only Lyapunov stability, i.e., no asymptotic escape from X^*).

Theorem 1 is modified from the Lyapunov stability theorem in Smirnov (2001, Theorem 8.2). While Smirnov's version is applicable to a singleton of an equilibrium point, the theorem here allows convergence to a set of equilibria.

There are a few differences from Zusai (2018, Theorem 7) to relax assumptions slightly. First, in the previous version, (5) is assumed for every point in the entire space $\mathcal{X}$ and $\tilde{W}$ is assumed to be continuous. By checking the places where the

definition of the domain for condition (i) in the theorem[10] and the continuity of $\tilde{W}$ are used in the proof of the previous version, one can easily find that it is innocuous to replace $\mathcal{X}$ with a closed subset X^* in a compact metric space and relax continuity of $\tilde{W}$ to lower semicontinuity.[11] Second but more importantly, the assumption in the previous version was a bit stronger condition than (5): every Carathéodory solution $\{\mathbf{x}_t\}$ starting from X' should satisfy

$$\dot{W}(\mathbf{x}_t) \leq \tilde{W}(\mathbf{x}_t) \quad \text{for almost all } t \in [0, \infty). \tag{S4}$$

If X' is forward invariant, then (5) implies this condition; thus, Zusai (2018) is straightforwardly applied and we can conclude that X^* is asymptotically stable and X' is a basin of attraction.

In part i) of Theorem 1 in the current version, we do not require forward invariance of X'; a solution trajectory may escape from A' and thus (S4) may not be maintained. The current version is more convenient to establish local stability, because we do not have to track a solution trajectory to see if (S4) is maintained.

Proof. Here we prove the difference in part i) from Zusai (2018, Theorem 7). For this, we focus on the case of $X' \subsetneq \mathcal{X}$ and find a forward invariant subset of X'. If we find it, any Carathéodory solution starting from the forward invariant subset remains there and thus satisfies (S4) as (5) holds for $\mathbf{x} = \mathbf{x}_t$ at each time $t \in \mathbb{R}_+$. Then, Zusai (2018, Theorem 7) is applied and assures asymptotic stability of X^* while having the forward invariant subset as a basin of attraction.

First, construct a distance from point $\mathbf{x} \in \mathcal{X}$ to X^* based on the metric on $\mathcal{X}$, say $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$, by

$$d_*(\mathbf{x}) \coloneqq \min_{\mathbf{x}^* \in X^*} d(\mathbf{x}, \mathbf{x}^*).$$

Since X^* is a non-empty compact set and $d(\mathbf{x}, \mathbf{x}^*)$ is continuous in $\mathbf{x}^*$ when $\mathbf{x}$ is fixed, Weierstrass theorem assures the existence of the minimum for any $\mathbf{x} \in X^*$. Thus, we have

$$d_*(\mathbf{x}) \geq 0; \qquad d_*(\mathbf{x}) = 0 \;\Leftrightarrow\; \mathbf{x} \in X^*.$$

[10] In the notation of the current version, the condition reads as $W(\mathbf{x}) \geq 0$ and $\tilde{W}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \mathcal{X}$. Thus it corresponds with condition (a) in Theorem 1.

[11] Specifically, $\check{A}$ in the proof (Zusai, 2018, p.25) should be defined as a subset of $\mathrm{cl}\, X'$. The continuity of $\tilde{W}$ was used to assure the existence of the minimum of $\tilde{W}$ in $\check{A}$; for this, lower semicontinuity is sufficient. Then, with the observation that forward invariance of X' implies (S4), the proof for the previous version applies to part ii) of the current version.

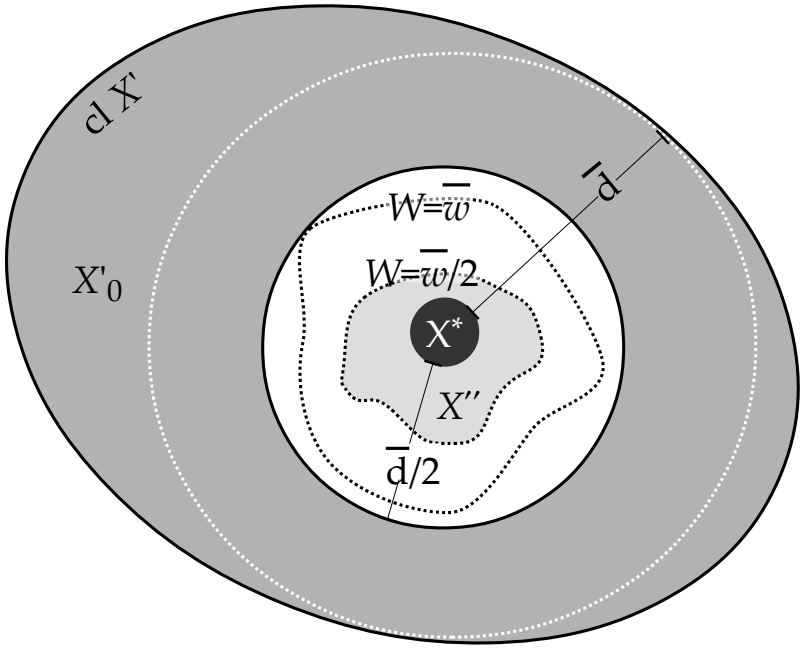


Figure S1: Sets in the proof of Theorem 1. X^* is the black area in the center and X'' is the light gray area. $\mathrm{cl}\, X'$ is the entire oval, with the outermost outline. X'_0 is the dark gray area, including the both boundaries.

Let $\bar{d}$ be the shortest distance from the complement of X' to X^*:

$$\bar{d} := \min_{\mathbf{x} \in \mathcal{X} \setminus X'} d_*(\mathbf{x}). \tag{S5}$$

Maximum theorem guarantees continuity of $d_* : \mathcal{X} \to \mathbb{R}_+$ by continuity of $d(\mathbf{x}, \mathbf{x}^*)$ in both $\mathbf{x}$ and $\mathbf{x}^*$. Besides, $\mathcal{X} \setminus X'$ is a non-empty compact subset by $X' \subsetneq \mathcal{X}$ and the openness of X'. Hence, the minimum in (S5) exists. It follows that

$$\bar{d} > 0; \qquad d_*(\mathbf{x}) < \bar{d} \;\Rightarrow\; \mathbf{x} \in X'. \tag{S6}$$

Define set $X'_0 \subset \mathrm{cl}\, X'$ by

$$X'_0 := \mathrm{cl}\, X' \;\cap\; {d_*}^{-1}([\bar{d}/2, \infty)).$$

Since both $\mathrm{cl}\, X'$ and ${d_*}^{-1}([\bar{d}/2, \infty))$ are closed, X'_0 is closed and thus compact in $\mathcal{X}$. It is not empty, as proven below. Suppose $X'_0 = \emptyset$; then, any $\mathbf{x} \in \mathcal{X}$ with $d_*(\mathbf{x}) \geq \bar{d}/2$ must be out of $\mathrm{cl}\, X'$. On the other hand, since $\mathrm{cl}\, X'$ is not empty, X' has

at least one boundary point $\mathbf{x}^0$; then, $d_*(\mathbf{x}^0) \geq \bar{d}$.[12] By the former statement, this implies $\mathbf{x}^0 \notin \operatorname{cl} X'$ but it contradicts with $\mathbf{x}^0$ being on the boundary of X'; hence, X'_0 cannot be empty.

Let $\bar{w}$ be the minimum of W in X'_0;

$$\bar{w} := \min_{\mathbf{x} \in X'_0} W(\mathbf{x}).$$

Since X'_0 is compact and nonempty and W is (Lipschitz) continuous, the minimum exists. Furthermore, it is positive; we have $X'_0 \subset \operatorname{cl} X'$ by construction and $W(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \operatorname{cl} X'$ by condition (a) and continuity of W, while no element $\mathbf{x} \in X'_0$ belongs to X^* since $d_*(\mathbf{x}) \geq \bar{d} > 0$ for any $\mathbf{x} \in X'_0$. Because $X^* = \operatorname{cl} X' \cap W^{-1}(0)$ by condition (b) and $X'_0 \subset \operatorname{cl} X'$, it implies $\mathbf{x} \in X'_0 \Rightarrow W(\mathbf{x}) > 0$. Hence we have $\bar{w} > 0$; by the definition of $\tilde{W}$, we have

$$\underbrace{\left[\mathbf{x} \in \operatorname{cl} X' \text{ and } d_*(\mathbf{x}) \geq \bar{d}/2\right]}_{\text{i.e., } \mathbf{x} \in X'_0} \Rightarrow\ W(\mathbf{x}) \geq \bar{w}. \tag{S7}$$

Define set $X'' \subset X'$ by

$$X'' = W^{-1}([0, \bar{w}/2)) \cap X'. \tag{S8}$$

This set is an (open) neighborhood of X^* by $X^* \subset X''$, since $W = 0$ at anywhere in X^* and $X^* \subset X'$. Now we prove that X'' is wholly contained in set $d_*^{-1}([0, \bar{d}/2))$. Assume that there exists $\mathbf{x} \in X''$ such that $d_*(\mathbf{x}) \geq \bar{d}/2$. These jointly imply $W(\mathbf{x}) \geq \bar{w}$ by (S7) since $\mathbf{x} \in X'' \subset X' \subset \operatorname{cl} X'$. However, this contradicts with $W(\mathbf{x}) \in [0, \bar{w}/2)$ for $\mathbf{x}$ to belong to X''. Hence, we have

$$\mathbf{x} \in X'' \ \Rightarrow\ d_*(\mathbf{x}) < \bar{d}/2. \tag{S9}$$

We prove X'' is forward invariant. To verify it by contradiction, assume that there is a Carathéodory solution trajectory $\{\mathbf{x}_t\}$ starting from X'' but escaping X'' at some moment of time:

$$\mathbf{x}_0 \in X'', \qquad \text{and } \mathbf{x}_T \notin X'' \text{ at some } T > 0. \tag{S10}$$

[12] We can make a sequence converging to $\mathbf{x}^0$ by elements out of X', whose distance from X^* cannot be smaller than $\bar{d}$.

The statement $\mathbf{x}_T \notin X''$ means $\mathbf{x}_T \notin X'$ or $W(\mathbf{x}_T) > \bar{w}/2$ by (S8). In the former case, we have $d_*(\mathbf{x}_T) \geq \bar{d}$ by (S5) while $d_*(\mathbf{x}_0) < \bar{d}/2$ by (S9). By continuity of $d_*(\mathbf{x})$ in $\mathbf{x}$ and of $\mathbf{x}_t$ in t on a Carathéodory solution trajectory $\{\mathbf{x}_t\}$, $d_*(\mathbf{x}_t)$ is continuous in t; hence, there exists a moment of time $T' \in (0, T)$ such that $d_*(\mathbf{x}_{T'}) = 0.9\bar{d} \in (0.5\bar{d}, \bar{d}) \subset (d_*(\mathbf{x}_0), d_*(\mathbf{x}_T))$. At this point, $\mathbf{x}_{T'} \notin X''$ by (S9) while $\mathbf{x}_{T'} \in X'$ by (S6); thus, $W(\mathbf{x}_{T'}) \geq \bar{w}/2$ by (S8). Hence, the first case of escaping X'' implies the existence of $T' > 0$ such that

$$W(\mathbf{x}_{T'}) \geq \bar{w}/2 \qquad \textit{and} \qquad \mathbf{x}_{T'} \in X'.$$

In the second (but not the first) case, we have $W(\mathbf{x}_T) \geq \bar{w}/2$ but $\mathbf{x}_T \in X'$; that is, the above statement holds with $T' = T$.

This implies the existence of $\bar{T} \in (0, T']$ such that

$$W(\mathbf{x}_{\bar{T}}) \geq \bar{w}/2, \qquad \text{and} \qquad \left[\mathbf{x}_t \in X' \text{ for all } t \leq \bar{T}\right]. \tag{S11}$$

To prove it, assume $\mathbf{x}_{t'} \notin X'$ at some $t' \leq T'$, i.e., the negation of the latter condition with $\bar{T} = T'$; if there is no such $t' \leq T'$, then it suggests that the claim (S11) holds at $\bar{T} = T'$ by the fact $W(\mathbf{x}_{T'}) \geq \bar{w}/2$. By (S6), the hypothesis $\mathbf{x}_{t'} \notin X'$ implies $d_*(\mathbf{x}_{t'}) \geq \bar{d}$. Again, by continuity of $d_*(\mathbf{x}_t)$ in t, the set $\{t \leq t' \mid d_*(\mathbf{x}_t) \geq \bar{d}\}$ is closed and thus compact; by the fact $d_*(\mathbf{x}_0) < \bar{d}/2$, this implies the existence of the minimum $\bar{T}$ in this set and $\bar{T} > 0$. That is, we have $d_*(\mathbf{x}_{\bar{T}}) = \bar{d}$ while $d_*(\mathbf{x}_t) < \bar{d}$ for all $t \leq \bar{T}$. The former implies $W(\mathbf{x}_{\bar{T}}) \geq \bar{w}$ by (S7) and the latter implies $\mathbf{x}_t \in X'$ for all $t < \bar{T}$ by (S6). Thus, the above claim (S11) holds at this $\bar{T} \in (0, T']$. Since condition (a) and (5) hold almost everywhere in X', we have $\dot{W}(\mathbf{x}_\tau) \leq \tilde{W}(\mathbf{x}_\tau) \leq 0$ at almost all $\tau \leq \bar{T}$;[13] thus, we have

$$W(\mathbf{x}_{\bar{T}}) \leq W(\mathbf{x}_0) + \int_0^{\bar{T}} \tilde{W}(\mathbf{x}_t)dt \leq W(\mathbf{x}_0).$$

Since $W(\mathbf{x}_0) < \bar{w}/2$ by $\mathbf{x}_0 \in X''$, we have $W(\mathbf{x}_{\bar{T}}) < \bar{w}/2$ in (S11). This contradicts with $W(\mathbf{x}_{\bar{T}}) \geq \bar{w}/2$.

Therefore, the hypothesis (S10) cannot hold: any Carathéodory solution trajectory $\{\mathbf{x}_t\}$ starting from X'' cannot escape X'' at any moment of time. That is, X'' is

[13] A Carathéodory solution trajectory is differentiable at almost all moments of time, though it may not be so at all moments.

forward invariant. □

S3 Supplement to Section 4

S3.1 The role of Assumption A1-ii)

From the proof of Lemma 1 in Appendix A1, we can see that the role of Assumption A1-ii) is to obtain (10) for property g1. Assumption A1-ii) implies that, given the preference ordering of actions based on the current payoff vector, for any $i \in \mathbb{N}$, the probability that (at least one of) the i-th best actions is available does not vary whether the current action is a or b, as long as the i-th action is better than both a and b; so, if a is worse than b, an agent who is currently taking a is more likely to switch than one currently taking b simply because the former action a has more actions (including b) that are better than a itself and thus the agent should switch to. Besides, if the former agent and the latter switch to the same action, the former improves the payoff more than the latter. Combining these two facts, the ex-ante gain by a switch from a should be greater than that from b. Therefore, the ordering of actions by the ex-ante gains is obtained by simply reversing the ordering by the current payoffs, as stated in property g1.[14]

Assumption A1-ii) may look a bit complicated. First, it checks the symmetry for each *set* $\mathcal{A}_{ab} \subset \mathcal{A} \setminus \{a, b\}$, not for each action $c \in \mathcal{A} \setminus \{a, b\}$. Second, it checks the probability of the event that *at least one action* in the set is available, not simply that the whole set is available. The first point assures invariance of correlations of any available actions other than a and b. The second point is made because we do not impose a condition on a set that contains either a or b—a set containing a cannot be assigned to a positive probability in $\mathbb{P}_{\mathcal{A}a}$ when a is the status quo.

One might think of somewhat simpler conditions that look similar to Assumption A1-ii):

i) For any $a, b \in \mathcal{A}$ and $c \in \mathcal{A} \setminus \{a, b\}$, $\mathbb{P}_{\mathcal{A}a}(\{\mathcal{A}'_a \mid c \in \mathcal{A}'_a\}) = \mathbb{P}_{\mathcal{A}b}(\{\mathcal{A}'_b \mid c \in \mathcal{A}'_b\})$. That is, the probability that *arbitrary action* c is available should not depend on the status quo, as long as strategy c is not the status quo.

[14] The switching cost does not matter to this and thus the monotone relationship as in property g1 is obtained as well for the gross gain.

ii) For any $a, b \in \mathcal{A}$ and $\mathcal{A}_{ab} \subset \mathcal{A} \setminus \{a, b\}$, $\mathbb{P}_{\mathcal{A}a}(\mathcal{A}_{ab}) = \mathbb{P}_{\mathcal{A}b}(\mathcal{A}_{ab})$. That is, for any subset that excludes both a and b, the probability that the available strategy set *coincides exactly* with this subset does not depend on which of the excluded two actions is the status quo.

However, neither of these two simpler conditions cannot assure (10). Each of the following cases satisfies one of the above two conditions but not Assumption A1-ii); as a result, (10) does not hold. When checking the violation of (10), payoff vector $\boldsymbol{\pi}$ is assumed to yield the payoff ranking such that $\pi_e > \pi_d > \pi_c > \pi_b > \pi_a$.

i) Assume that, when a is the status quo, then the available strategy set is $\{b\}$ with probability 0.4, $\{c, d\}$ with 0.3 and $\{e\}$ with 0.3. When b is the status quo, then the available strategy set is $\{a\}$ with probability 0.4, $\{c\}$ with 0.3 and $\{d, e\}$ with 0.3.

Condition i) is satisfied, since each of the three actions c, d and e is available with probability 0.3, whether the status quo is a or b. However, Assumption A1-ii) is not satisfied since at least *d or e* is available with probability $\mathbb{P}_{\mathcal{A}a}(\{\mathcal{A}'_a \mid \mathcal{A}'_a \cap \{d, e\} \neq \emptyset\}) = \mathbb{P}_{\mathcal{A}a}(\{\{c, d\}\}) + \mathbb{P}_{\mathcal{A}a}(\{\{e\}\}) = 0.6$ when a is the status quo while this probability is $\mathbb{P}_{\mathcal{A}b}(\{\mathcal{A}'_b \mid \mathcal{A}'_b \cap \{d, e\} \neq \emptyset\}) = \mathbb{P}_{\mathcal{A}a}(\{\{d, e\}\}) = 0.3$.

Note that, if the second best d is available, the first best e is not available when a is the status quo; when b is the status quo, e is also available. As a result, the probability that the second best d is the best available strategy is $p^2_{\mathcal{A}a} = 0.3$ when a is the status quo, while it is $p^2_{\mathcal{A}b} = 0$ when b is the status quo. Thus, (10) does not hold. This is because condition i) allows correlation of availabilities of distinct two strategies to vary with the status quo.

ii) Assume that, when a is the status quo, then the available strategy set is $\{b\}$ with probability 0.8, $\{b, c, d\}$ with 0.1, and $\{b, e\}$ with 0.1. When b is the status quo, then the available strategy set is $\{a\}$ with probability 0.8, $\{a, c\}$ with 0.1, and $\{a, d, e\}$ with 0.1.

Condition ii) is trivially satisfied, since no subset that excludes both a and b is available with any positive probability when either a or b is the status quo. (Note that condition i) is also satisfied.) However, Assumption A1-ii) is not satisfied, since at least *d or e* is available with probability 0.2 when a is the status quo; this probability is 0.1 when b is the status quo.

Similarly to the counterexample for condition i), $\mathbb{P}_{\mathcal{A}a}$ and $\mathbb{P}_{\mathcal{A}b}$ have different correlations between availabilities of the first best action e and the second best action d; this results in $p^2_{\mathcal{A}a} = 0.1 \neq 0 = p^2_{\mathcal{A}b}$.

S4 Supplement to Section 5

S4.1 Supplement to Section 5.1

Modified framework for excess payoff dynamics

To interpret excess payoff dynamics as economically reasonable, we need just one modification of our framework: the status-quo strategy should be mixed strategy $\mathbf{x}$, i.e., the average strategy (action distribution) of agents in the population. We imagine a birth-death process over generations (Alós-Ferrer and Neustadt, 2010), rather than a revision process of immortal agents as in the basic framework.[15] Everything else is the same as the basic framework. If the agent chooses to switch an action, then it chooses a pure strategy of a particular available action and pays switching cost q. Available action set $\mathcal{A}' \subset \mathcal{A}$ is randomly drawn from probability distribution $\mathbb{P}_{\mathcal{A}}$ over the power set of $\mathcal{A}$; switching cost q follows probability distribution $\mathbb{P}_Q$ over $\mathbb{R}_+$. A newborn agent compares the net payoffs from pure strategies of available actions with the expected payoff from the default mixed strategy $\mathbf{x}$; then the newborn agent decides whether to adopt the default strategy $\mathbf{x}$ or to pay cost q and take a pure strategy of an available action.

On $\mathbb{P}_Q$, we maintain Assumption Q1. On $\mathbb{P}_{\mathcal{A}}$, we modify Assumption A1-i):

Assumption (A1-i)'**).** For any $b \in \mathcal{A}$, there is a set $\mathcal{A}' \subset \mathcal{A}$ such that $\mathbb{P}_{\mathcal{A}}(\mathcal{A}') > 0$ and $b \in \mathcal{A}'$.

Assumption A1-ii) does not make sense for this situation and thus we drop it.

The (gross) payoff gain from a switch to action a is indeed the excess payoff of the action, i.e., $\hat{\pi}_a := \pi_a - \boldsymbol{\pi} \cdot \mathbf{x}$; let $\hat{\pi}_*(\mathcal{A}') := \pi_*(\mathcal{A}') - \boldsymbol{\pi} \cdot \mathbf{x}$ Thus, after drawing available action set $\mathcal{A}'$, an agent's conditional switching rate must belong to $\mathcal{Q}(\hat{\pi}_*(\mathcal{A}'))$.

[15] Specifically, an agent's death follows a Poisson process with arrival rate 1. At the same time as one's death, a new agent is born. Unless a newborn agent chooses to pay q, the agent is assigned to a randomly chosen action, drawn from the population's action distribution.

Thus, the transition of the social state follows

$$\dot{\mathbf{x}} \in \mathbf{V}(\mathbf{x})[\boldsymbol{\pi}] = \sum_{\mathcal{A}' \subset \mathcal{A}} \mathbb{P}_{\mathcal{A}}(\mathcal{A}') \mathcal{Q}(\hat{\pi}_*(\mathcal{A}')) \left(\Delta^{\mathcal{A}}(b_*(\mathcal{A}')[\boldsymbol{\pi}] - \mathbf{x} \right).$$

We call an evolutionary dynamic **a birth-death economically reasonable dynamic** if it is constructed in this way and satisfies Assumptions Q1 and A1-i). It includes excess payoff dynamics and also some new dynamics.

Example S6 (Excess payoff dynamics). Let $\mathbb{P}_{\mathcal{A}}(\{a\}) = 1/A$ for each $a \in \mathcal{A}$: a newborn agent randomly samples an action from $\mathcal{A}$, like pairwise payoff comparison dynamics (Example S2). So, upon the receipt of a revision opportunity, the probability that an agent eventually chooses action a is simply $\mathcal{Q}(\hat{\pi}_a)$. This is an excess payoff dynamic. Brown-von Neumman-Nash dynamic is obtained by $Q(\hat{\pi}) = [\hat{\pi}]_+$, which is generalized to $Q(\hat{\pi}) = ([\hat{\pi}]_+)^k$ for (k-)BNN dynamic: (Sandholm, 2010b, Exercise 5.5.1). As $k \to \infty$, the switch to the best response dominates all the other switches and thus the behavior of the dynamic should look like the standard BRD. ∎

Example S7 (Nontrivial restriction on available actions). We could allow a non-trivial restriction to the available action set, just as Example S3, while setting a status quo strategy to $\mathbf{x}$. ∎

Example S8 ("Tempered" BRD without hysteresis). Let $\mathbb{P}_{\mathcal{A}}(\mathcal{A}) = 1$ (i.e., all actions are always available) and $Q(\hat{\pi})$ being an increasing function of $\hat{\pi}$, just like the tempered BRD (Example S1). However, in this excess payoff framework, an agent who chooses the status quo take $\mathbf{x}$ and thus any hysteresis of the agent's past action disappears. Thus, unlike the tempered BRD, the tempering function Q only changes the speed of the dynamic from the standard BRD. The trajectory itself is exactly the same as that under the standard BRD. ∎

Equilibrium stability

Now we can define an individual's first-order gain $g_*(\mathbf{x})[\boldsymbol{\pi}]$ as

$$g_*(\mathbf{x})[\boldsymbol{\pi}] := \sum_{\mathcal{A}' \subset \mathcal{A}} \mathbb{P}_{\mathcal{A}}(\mathcal{A}') \mathbb{E}_Q[\hat{\pi}_*(\mathcal{A}') - q]_+.$$

Note that it depends on $\mathbf{x}$ as well as $\boldsymbol{\pi}$, but all the agents share the same gain regardless of their current actions. As a result, the second order gain is zero; were

Birth-death economically reasonable dynamics

$\mathbb{P}_{\mathcal{A}}$: Assumption A1-i)′; Q: Assumption Q1.

- $\mathbf{V}(\mathbf{x})[\boldsymbol{\pi}] = \sum_{\mathcal{A}' \subset \mathcal{A}} \mathbb{P}_{\mathcal{A}}(\mathcal{A}') Q(\hat{\pi}_*(\mathcal{A}')) \left(\Delta^{\mathcal{A}}(b_*[\boldsymbol{\pi}; \mathcal{A}'] - \mathbf{x} \right).$
- $G(\mathbf{x})[\boldsymbol{\pi}] = \sum_{\mathcal{A}' \subset \mathcal{A}} \mathbb{P}_{\mathcal{A}}(\mathcal{A}') \mathbb{E}_Q[\hat{\pi}_*(\mathcal{A}') - q]_+.$
- $H(\mathbf{x})[\boldsymbol{\pi}] = -\left\{ \sum_{\mathcal{A}' \subset \mathcal{A}} \mathbb{P}_{\mathcal{A}}(\mathcal{A}') Q(\hat{\pi}_*(\mathcal{A}')) \right\} \left\{ \sum_{\mathcal{A}' \subset \mathcal{A}} \mathbb{P}_{\mathcal{A}}(\mathcal{A}') Q(\hat{\pi}_*(\mathcal{A}')) \hat{\pi}_*(\mathcal{A}') \right\}.$

Excess payoff dynamics (Hofbauer and Sandholm, 2009)

$\mathbb{P}_{\mathcal{A}}(\{b\}) = 1/A$ for each $b \neq a$; Q: any function that meets Assumption Q1.

- $\mathbf{V}(\mathbf{x})[\boldsymbol{\pi}] = \sum_{b \in \mathcal{A}} \frac{Q(\hat{\pi}_b)}{A} (\mathbf{e}_b - \mathbf{x}).$
- $G(\mathbf{x})[\boldsymbol{\pi}] = \sum_{b \in \mathcal{A}} \frac{1}{A} \mathbb{E}_Q[\hat{\pi}_b - q]_+.$
- $H(\mathbf{x})[\boldsymbol{\pi}] = -\left\{ \sum_{b \in \mathcal{A}} \frac{Q(\hat{\pi}_b)}{A} \right\} \left\{ \sum_{b \in \mathcal{A}} \frac{Q(\hat{\pi}_b)}{A} \hat{\pi}_b \right\}.$

Brown-von Neuman-Nash dynamic (Hofbauer, 2001)

$\mathbb{P}_{\mathcal{A}}(\{b\}) = 1/A$ for each $b \neq a$; $Q(q) = [q]_+$.

- $\mathbf{V}(\mathbf{x})[\boldsymbol{\pi}] = \sum_{b \in \mathcal{A}} \frac{[\hat{\pi}_b]_+}{A} (\mathbf{e}_b - \mathbf{x}).$
- $G(\mathbf{x})[\boldsymbol{\pi}] = \sum_{b \in \mathcal{A}} \frac{1}{2A} [\hat{\pi}_b - q]_+^2.$
- $H(\mathbf{x})[\boldsymbol{\pi}] = -\left\{ \sum_{b \in \mathcal{A}} \frac{[\hat{\pi}_b]_+}{A} \right\} \left\{ \sum_{b \in \mathcal{A}} \frac{[\hat{\pi}_b]_+}{A} \hat{\pi}_b \right\}.$

Figure S2: Major dynamics captured under the modified framework.

the payoff vector fixed constant, we could say that the choice of action in the first revision does not change the gain in the second revision.[16] With the total mass of agents equal to one, the aggregate first-order gain $G(\mathbf{x})[\boldsymbol{\pi}]$ is now just equal to the individual first-order gain: $G(\mathbf{x})[\boldsymbol{\pi}] = g_*(\mathbf{x})[\boldsymbol{\pi}] \cdot 1$. Similar to Theorem 2, we can prove G satisfies properties G and GH-ii).

While the aggregate second-order gain cannot be used as a decay rate function H, we find that the following function H works as the decay rate function coupled with G as a Lyapunov function to meet properties in Theorem 2. This function H is obtained from $\partial G/\partial \mathbf{x}$ to meet property GH-i).

$$H[\boldsymbol{\pi}](\mathbf{x}) = -\left\{\sum_{\mathcal{A}'\subset\mathcal{A}} \mathbb{P}_{\mathcal{A}}(\mathcal{A}')Q(\hat{\pi}_*(\mathcal{A}'))\right\}\left\{\sum_{\mathcal{A}'\subset\mathcal{A}} \mathbb{P}_{\mathcal{A}}(\mathcal{A}')Q(\hat{\pi}_*(\mathcal{A}'))\hat{\pi}_*(\mathcal{A}')\right\}.$$

From this form, it is immediate to see that H satisfies property H as well. Therefore, the pair of these G and H assures δ-passivity of birth-death economically reasonable dynamics.

S4.2 Supplement to Section 5.3

Set-up of a multi-population game.

The society consists of a unit mass of agents, each of whom belongs to either one of P populations; let $m^p > 0$ be the mass of population $p \in \mathcal{P}$ and $\sum_{p\in\mathcal{P}} m^p = 1$. In each population $p \in \mathcal{P}$, all the agents share the same action set $\mathcal{A}^p = \{1, \ldots, A^p\}$. We denote by x_a^p the mass of action-a agents in population p; population p's state $\mathbf{x}^p := (x_a^p)_{a\in\mathcal{A}^p}$ belongs to $\mathcal{X}^p := m^p\Delta^{\mathcal{A}^p}$. The social state $\mathbf{x}^{\mathcal{P}} := (\mathbf{x}^p)_{p\in\mathcal{P}} \in \mathbb{R}^{A^{\mathcal{P}}}$ is a collection of each population's state. Here, $A^{\mathcal{P}} := \sum_{p\in\mathcal{P}} A^p$. Denote by $\mathcal{X}^{\mathcal{P}} := \times_{p\in\mathcal{P}} \mathcal{X}^p$ the set of all the feasible social states.

Agents in the same population evaluates the social state based on the same payoff function $\mathbf{F}^p : \mathcal{X}^{\mathcal{P}} \to \mathbb{R}^{A^p}$. $\mathbf{F}^p(\mathbf{x}^{\mathcal{P}})$ is the payoff vector for agents in population $p \in \mathcal{P}$ when the social state is $\mathbf{x}^{\mathcal{P}}$. We denote by $\boldsymbol{\pi}^{\mathcal{P}}$ the collection of payoff vectors $\boldsymbol{\pi}^p \in \mathbb{R}^{A^p}$ over all the populations $p \in \mathcal{P}$: i.e., $\boldsymbol{\pi}^{\mathcal{P}} = (\boldsymbol{\pi}^p)_{p\in\mathcal{P}} \in \mathbb{R}^{A^{\mathcal{P}}}$.[17] The definition of a contractive game (Definition S1) is extended to a multi-population setting

[16] In the interpretation as a birth-death process, we could simply say that mortal agents do not have the second revision.

[17] Both $\mathbf{x}^{\mathcal{P}}$ and $\boldsymbol{\pi}^{\mathcal{P}}$ are $A^{\mathcal{P}}$-dimensional column vectors.

by requiring $\mathbf{F}, \mathbf{x}, \mathbf{y}$ in condition (S1) to be replaced with $\mathbf{F}^{\mathcal{P}}, \mathbf{x}^{\mathcal{P}}, \mathbf{y}^{\mathcal{P}}$. With similar replacement, the definition of static stability (Definition 1) and the equivalent first-order condition (S2) are extended to a multi-population contractive game.

Example S9 (Saddle games). It has been long to call game theorists' attention to a multi-player game with a saddle function $\phi : \mathcal{X}^{\mathcal{P}} \to \mathbb{R}$ that is concave in $\mathbf{x}^C := (\mathbf{x}^p)_{p \in \mathcal{P}^C}$ and convex in $\mathbf{x}^V := (\mathbf{x}^p)_{p \in \mathcal{P}^V}$.[18] Here, the population set is partitioned to two distinct, nonempty subsets $\mathcal{P}^C$ and $\mathcal{P}^V$. We define the game as a population game where population p's payoff function is given by

$$\mathbf{F}^p(\mathbf{x}^{\mathcal{P}}) := \begin{cases} \frac{\partial \phi}{\partial \mathbf{x}^c}(\mathbf{x}^{\mathcal{P}}) & \text{if } p = c \in \mathcal{P}^C, \\ -\frac{\partial \phi}{\partial \mathbf{x}^v}(\mathbf{x}^{\mathcal{P}}) & \text{if } p = v \in \mathcal{P}^V. \end{cases}$$

When population $c \in \mathcal{P}^C$ takes the best response $B^c(\mathbf{x}^{\mathcal{P}})$, it maximizes function ϕ with respect to $\mathbf{x}^c$ given the current $\mathbf{x}^{\mathcal{P}}$; population $v \in \mathcal{P}^V$ minimizes ϕ in the best response. Assuming second-order continuous differentiability of ϕ, we have negative semidefiniteness of $\frac{\partial \mathbf{F}^C}{\partial \mathbf{x}^C}(\mathbf{x}^{\mathcal{P}})$ and $\frac{\partial \mathbf{F}^V}{\partial \mathbf{x}^V}(\mathbf{x}^{\mathcal{P}})$ as well as $\frac{\partial \mathbf{F}^V}{\partial \mathbf{x}^C} + \frac{\partial \mathbf{F}^C}{\partial \mathbf{x}^V} \equiv \frac{\partial^2 \phi}{\partial \mathbf{x}^C \partial \mathbf{x}^V} - \frac{\partial^2 \phi}{\partial \mathbf{x}^V \partial \mathbf{x}^C} \equiv 0$ by Young's theorem. These jointly imply that $\mathbf{F}^{\mathcal{P}}$ is a contractive game. Nash equilibrium $\mathbf{x}^{\mathcal{P}*}$ is a saddle point of ϕ. ∎

Example S10 (Aggregate games). Although different populations are distinguished differently in the above presentation of the model, they need not to be. In an **aggregate game**, all populations $p \in \mathcal{P}$ share the same action set $\mathcal{A}^p = \mathcal{A}$ and each population's payoff vector depends only on the aggregate action distribution over the society $\bar{\mathbf{x}} := \sum_{q \in \mathcal{P}} \mathbf{x}^q \in \Delta^{\mathcal{A}}$, but not on the profile of those in each population $\mathbf{x}^{\mathcal{P}}$. We assume that payoff heterogeneity is additively separable: each population's payoff function $\mathbf{F}^p : \mathcal{X}^{\mathcal{P}} \to \mathbb{R}^A$ consists of common payoff function $\mathbf{F}^0 : \Delta^{\mathcal{A}} \to \mathbb{R}^A$ and constant payoff perturbation $\boldsymbol{\theta}^p \in \mathbb{R}^A$ as $\mathbf{F}^p(\mathbf{x}^{\mathcal{P}}) = \mathbf{F}^0(\sum_{q \in \mathcal{P}} \mathbf{x}^q) + \boldsymbol{\theta}^p$. If the common payoff function $\mathbf{F}^0$ is a contractive game in the homogeneous setting, then the aggregate game constructed in this way is also a contractive game. ∎

[18] Classical stability results by Kose (1956) can be interpreted as showing asymptotic stability of a unique Nash equilibrium under excess payoff dynamics in a game with a strict saddle function. Interestingly, Kose considers an application to numerical solution of a saddle value problem and proposes an electro-analog computer for computation. Nora and Uno (2014) relate correlated equilibria in a saddle game to robust equilibria under incomplete information. Sorin and Hofbauer (2005) verify Nash stability in this class of games under the standard BRD, while they regard this game as a zero-sum game. A class of saddle games is broader than that of zero-sum games; see an example of private provision of defense services in Nora and Uno (2014, Example 1).

Evolutionary dynamics and gains

Let $\mathbf{V}^p : \mathcal{X}^p \times \mathbb{R}^{A^p} \to T\mathcal{X}^p$ be the evolutionary dynamic for population $p \in \mathcal{P}$, which determines the transition of population p's state as $\dot{\mathbf{x}}^p \in \mathbf{V}^p(\mathbf{x}^p)[\boldsymbol{\pi}^p]$. We allow each population p to follow a different revision protocol but we assume that $\mathbf{V}^p$ depends only on $\mathbf{x}^p$ and $\boldsymbol{\pi}^p$.[19] Then, the social dynamic of $\mathbf{x}^{\mathcal{P}}$ is constructed simply as

$$\dot{\mathbf{x}}^{\mathcal{P}} = (\dot{\mathbf{x}}^p)_{p\in\mathcal{P}} \in \mathbf{V}^{\mathcal{P}}(\mathbf{x}^{\mathcal{P}})[\boldsymbol{\pi}^{\mathcal{P}}] := \bigtimes_{p\in\mathcal{P}} \mathbf{V}^p(\mathbf{x}^p)[\boldsymbol{\pi}^p].$$

We assume that each population's evolutionary dynamic $\mathbf{V}^p$ has functions $G^p, H^p : \Delta^{\mathcal{A}^p} \times \mathbb{R}^{A^p} \to \mathbb{R}$ that satisfy properties G, H and GH in Theorem 2.[20] Then, we can add up those functions over all the populations to define functions $G^{\mathcal{P}}, H^{\mathcal{P}} : \mathcal{X}^{\mathcal{P}} \times \mathbb{R}^{A^{\mathcal{P}}} \to \mathbb{R}$ for the entire society $\mathcal{P}$ by

$$G^{\mathcal{P}}(\mathbf{x}^{\mathcal{P}})[\boldsymbol{\pi}^{\mathcal{P}}] := \sum_{p\in\mathcal{P}} G^p(\mathbf{x}^p)[\boldsymbol{\pi}^p], \quad \text{and} \quad H^{\mathcal{P}}(\mathbf{x}^{\mathcal{P}})[\boldsymbol{\pi}^{\mathcal{P}}] := \sum_{p\in\mathcal{P}} H^p(\mathbf{x}^p)[\boldsymbol{\pi}^p].$$

As these are simply sums, it is easy to derive properties G, H and GH of $G^{\mathcal{P}}$ and $H^{\mathcal{P}}$ from those of G^p and H^p over all populations $p \in \mathcal{P}$.[21]

S5 Supplement to Section 6

S5.1 Supplement to Section 6.1

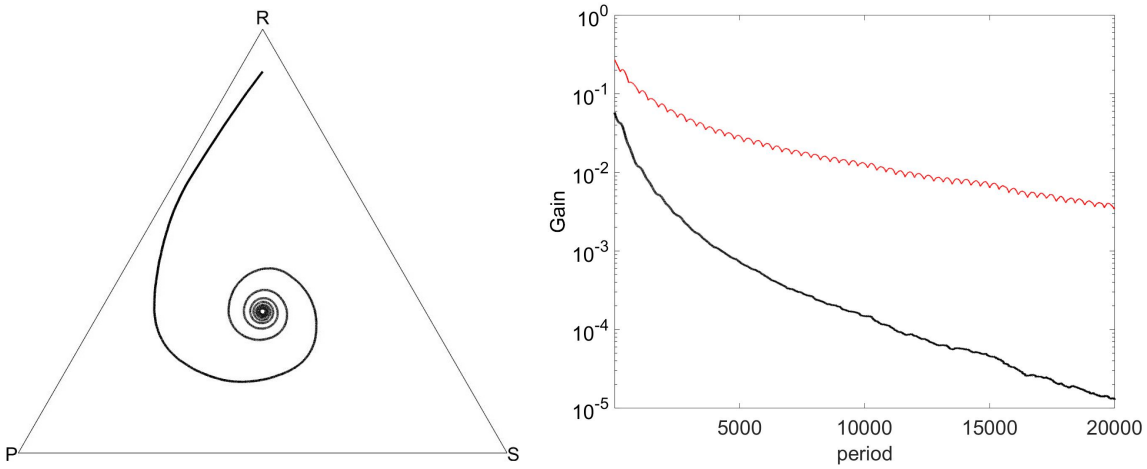
In the main text, we show possibility that the aggregate *gross* gain *may* not decrease over time even if $\mathbf{F}$ satisfies static stability and $\mathbf{V}$ is economically reasonable. Below we verify this possibility by showing an example.

Example S11 (Rock-Paper-Scissors). Here we consider random matching in a "good" Rock-Paper-Scissors game (Sandholm, 2010b, Example 3.3.2), where a player gets payoff of 1 from a win, 0 from a draw and -0.9 from a loss. This game is strictly

[19] For non-observational dynamics, it only requires uncoupledness—independence of the dynamic from $\boldsymbol{\pi}^{-p}$. However, for observational dynamics, it requires an agent's observations of others' choices to be limited to the same population.

[20] These functions are not necessarily derived as first/second-order gains. We only need those properties.

[21] The set of mixed strategies over optimal actions $\Delta^{\mathcal{A}}(b_*[\boldsymbol{\pi}])$ should be redefined as $\times_{p\in\mathcal{P}} \Delta^{\mathcal{A}^p}(b_*[\boldsymbol{\pi}^p])$ when Theorem 2 is read for $G^{\mathcal{P}}$ and $H^{\mathcal{P}}$.



(a) Trajectory under Smith

(b) Net and aggregate gains under Smith

Figure S3: Convergence to the unique Nash equilibrium in a good RPS and changes in the aggregate gross and net gains under a pairwise comparison dynamic (Smith). In Figure S3b, the aggregate net gain G is shown with a black thick curve and the aggregate gross gain Γ with a red thin curve.

contractive and thus the unique Nash equilibrium $\mathbf{x} = (x_R, x_P, x_S) = (1/3, 1/3, 1/3)$ is globally asymptotically stable under economically reasonable dynamics. In Figure S3a, the Smith dynamic starts from $\mathbf{x} = (0.9, 0.05, 0.05)$ and indeed converges to $(1/3, 1/3, 1/3)$. In Figure S3b, the aggregate net gain G decreases over time while the aggregate gross gain Γ has a decreasing trend but with perpetual fluctuations. ∎

S5.2 Supplement to Section 6.2

Since our economically reasonability guarantees local stability of a regular ESS while order compatibility in Friedman (1991) does not, it would be immediate to see that order compatibility does not imply our economic reasonability. But, why is that so? For this, consider the example that was used in his paper to show instability under order compatible dynamics.

Example S12 (Friedman 1991, Counterexample 2). Consider a linear one-population

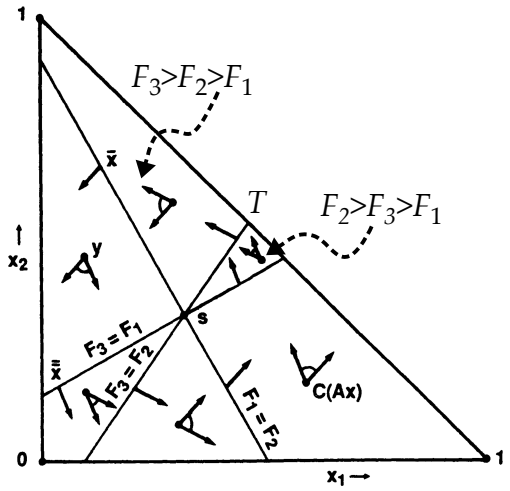


Figure S4: Phase diagram of order compatible dynamics in Example S12, borrowed from Friedman (1991, Figure 7). The labels of the payoff orderings and point T (those in roman) are added to the original. The solid small arrows on the original figure indicate the possible ranges of transition vectors under order compatible dynamics.

game $\mathbf{F} : \Delta^3 \to \mathbb{R}^3$ given by

$$\mathbf{F}(\mathbf{x}) := \begin{pmatrix} -5 & -26 & 31 \\ 34 & -5 & -29 \\ -29 & 31 & -2 \end{pmatrix} \mathbf{x}.$$

This game has a unique Nash equilibrium at $\mathbf{x}^* = (1/3, 1/3, 1/3)$. The game is indeed strictly contractive and $\mathbf{x}^*$ is a regular ESS.

One can construct an order compatible dynamic that does not converge to $\mathbf{x}^*$ "by always picking tangent vectors [from the cone of possible transition vectors that order compatibility permits at each point of the state] close to the outer edge of the cones [of those permissible transition vectors]" (p.655). Call such a dynamic a *diverging extreme dynamic*. For example, consider an interior state $\mathbf{x}$ where $F_3(\mathbf{x}) > F_2(\mathbf{x}) > F_1(\mathbf{x})$. Order compatibility requires transition vector $\mathbf{z}$ to satisfy $z_3 > z_2 > z_1$. The transition vector in a diverging extreme dynamic should particularly meet $z_3 \approx z_2 > 0 > z_1$. Thus, such a dynamic should assign to the second best action 2 so a large net inflow as to the first best action 3.

A diverging extreme dynamic needs to favor the *second best* response action at each state almost as much as the first best though the second best action (or the

payoff ordering) may change as the state $\mathbf{x}$ moves, while our framework (especially, Assumption A0 and independence of the availability of each action from payoff ordering of the actions) prohibits $\mathbb{P}_{\mathcal{A}1}$ from favoring one action than another in one state but opposing it in the other state. On the other hand, economically reasonable dynamics may not be order compatible and the reason can be found in the distribution of available action sets.

In the following, we confirm that neither Friedman's order compatibility nor our economic reasonability imply each other.

Order compatibility $\not\Rightarrow$ economic reasonability. Under an economic reasonable dynamic with continuous $\mathbb{P}_Q$, a transition vector $\mathbf{z}$ should take a form such as

$$\begin{aligned} z_2 &= x_1\mathbb{P}_{\mathcal{A}_1}(\{2\})Q(F_2(\mathbf{x})-F_1(\mathbf{x})) && \overbrace{-x_2\mathbb{P}_{\mathcal{A}_2}(\{\mathcal{A}_2 \mid 3\in\mathcal{A}_2\})Q(F_3(\mathbf{x})-F_2(\mathbf{x}))}^{\text{gross flow from 2 to 3}}, \\ z_3 &= x_1\mathbb{P}_{\mathcal{A}_1}(\{\mathcal{A}_1 \mid 3\in\mathcal{A}_1\})Q(F_3(\mathbf{x})-F_1(\mathbf{x})) && +x_2\mathbb{P}_{\mathcal{A}_2}(\{\mathcal{A}_2 \mid 3\in\mathcal{A}_2\})Q(F_3(\mathbf{x})-F_2(\mathbf{x})). \end{aligned} \tag{S12}$$

The gross flow from action 2 to action 3 (the identical second terms *after* the minus/plus signs in the two equations) is (strictly) positive by $x_2 > 0$ and Assumptions Q1 and A1-i). Beside, since Q is non-decreasing, we have $Q(F_3(\mathbf{x})-F_1(\mathbf{x})) \geq Q(F_2(\mathbf{x})-F_1(\mathbf{x}))$. Therefore, $z_3 = z_2$ holds only if

$$\mathbb{P}_{\mathcal{A}1}(\{2\}) > \mathbb{P}_{\mathcal{A}1}(\{\mathcal{A}_1 \mid 3\in\mathcal{A}_1\}).$$

In other words, for an agent who has taken action 1, it must be more likely to have action 2 available but 3 not available than to have action 3 available:

Now, consider an inner state where $F_2(\mathbf{x}) > F_3(\mathbf{x}) > F_1(\mathbf{x})$; the transition vector $\mathbf{z}$ in a diverging extreme dynamic should satisfy $z_2 = z_3 > 0 > z_1$. By the same token as above, it must be the case for economic reasonability of $z_2 = z_3$ that an agent who has taken action 1 is now more likely to have action 3 available but 1 not than having action 2 available:

$$\mathbb{P}_{\mathcal{A}1}(\{3\}) > \mathbb{P}_{\mathcal{A}1}(\{\mathcal{A}_1 \mid 2\in\mathcal{A}_1\}).$$

Since $\mathbb{P}_{\mathcal{A}1}(\{b\}) \leq \mathbb{P}_{\mathcal{A}1}(\{\mathcal{A}_1 \mid b\in\mathcal{A}_1\})$ for any $b \in \mathcal{A}\setminus\{1\} = \{2,3\}$, the above two equations contradict each other.

Economic reasonability $\not\Rightarrow$ order compatibility. Now consider a point where $F_3(\mathbf{x}) > F_2(\mathbf{x}) > F_1(\mathbf{x})$. According to (S12), an economically reasonable dynamic may have $z_2 > z_3$ and thus violate order compatibility if i) $\mathbb{P}_{\mathcal{A}1}(\{2\}) \gg \mathbb{P}_{\mathcal{A}1}(\{\mathcal{A}_1 \mid 3 \in \mathcal{A}_1\})$ and ii) $Q(F_3(\mathbf{x}) - F_1(\mathbf{x})) \approx Q(F_2(\mathbf{x}) - F_1(\mathbf{x})) \gg Q(F_3(\mathbf{x}) - F_2(\mathbf{x})) \approx 0$. Below we concretely find economically reasonable dynamics that satisfy these two conditions.

First, we can make $\mathbb{P}_{\mathcal{A}\cdot}$ to satisfy condition i) as well as Assumptions A1-i), A1-ii) and A0: for example,

$$\begin{aligned} &\mathbb{P}_{\mathcal{A}1}(\{2\}) = 3/4, && \mathbb{P}_{\mathcal{A}1}(\{3\}) = 1/4; \\ &\mathbb{P}_{\mathcal{A}2}(\{1\}) = 3/4, && \mathbb{P}_{\mathcal{A}2}(\{1,3\}) = 1/4; \\ &\mathbb{P}_{\mathcal{A}3}(\{1,2\}) = 3/4, && \mathbb{P}_{\mathcal{A}3}(\{1\}) = 1/4. \end{aligned}$$

Note that, regardless of the current action, action 1 is available with probability 1, action 2 with $3/4$ and action 3 with $1/4$ as long as the action is not the current action.

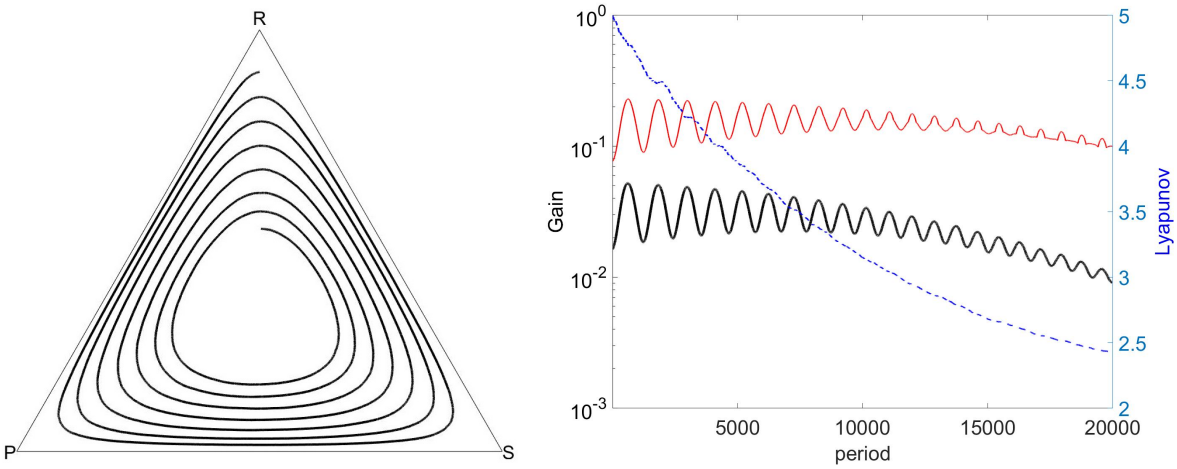
To meet condition ii), focus on an inner state $\mathbf{x}$ near the boundary $F_3 = F_2$ but a little away from $F_2 = F_1$, i.e., near the intersection of $F_3 = F_2$ and $x_1 + x_2 = 1$ (i.e., $x_3 = 0$); see point T on Figure S4. While satisfying Assumption Q1, let function $Q(q)$ be continuous (so $\mathcal{Q}(q) \equiv \{Q(q)\}$) and strictly increasing until q increases to $F_3(\mathbf{x}) - F_2(\mathbf{x}) \approx 0$ but let it be constant when q is between $F_2(\mathbf{x}) - F_1(\mathbf{x})$ and $F_3(\mathbf{x}) - F_1(\mathbf{x})$.

The economically reasonable dynamic with such $\mathbb{P}_{\mathcal{A}\cdot}$ and Q fails order compatibility but still guarantees Nash stability in this game and thus (not only local but) global stability of the unique regular ESS. ∎

S5.3 Supplement to Section 6.3

In the main text, we show possibility that the aggregate net gain *may* not decrease over time under imitative dynamics even if $\mathbf{F}$ satisfies static stability. Below we verify this possibility by showing an example.

Example S13 (Rock-Paper-Scissors again). Again, we consider random matching in the good Rock-Paper-Scissors as in Example S11. Since it is a strictly contractive game, the unique Nash equilibrium $\mathbf{x}^* = (1/3, 1/3, 1/3)$ is globally asymptotically stable under the replicator dynamic as well as economically reasonable dynamics. Figure S5 illustrates the evolution under the replicator dynamic. Both the aggregate net gain G and the aggregate gross gain Γ fluctuate. According to Taylor and Jonker



(a) Trajectory under the replicator dynamic (b) Net and aggregate gains under the replicator dynamic

Figure S5: Convergence to the unique Nash equilibrium in a good RPS and changes in the aggregate gross and net gains under the replicator dynamic. In Figure S5b, the known Lyapunov function $W_{\mathbf{x}^*}(\mathbf{x})$ is shown with a blue dashed curve, the aggregate net gain G with a black thick solid curve and the aggregate gross gain with a red thin solid curve.

(1978), function $W_{\mathbf{x}^*}(\mathbf{x}) = \sum_{a\in\mathcal{A}}(1/3)(\ln x_a^* - \ln x_a)$ works as a Lyapunov function. Actually it decreases over time; the trajectory converges from $(0.9, 0.05, 0.05)$ to $\mathbf{x}^*$. ∎

Remark. Mertikopoulos and Sandholm (2018) provide a general form of the Lyapunov function for global stability of the Nash equilibrium set in a strictly contractive game and local stability of an ESS under "Riemannian" game dynamics, including these two dynamics as canonical examples. For the replicator dynamic, the general Lyapunov function reduces to $W_{\mathbf{x}^*}(\mathbf{x}) = \sum_{a\in S(\mathbf{x}^*)} x_a^* \ln(x_a^*/x_a)$, as in Hofbauer et al. (1979); Zeeman (1980). Here $S(\mathbf{x}^*) \subset \mathcal{A}$ is the set of actions used in $\mathbf{x}^*$. Those known Lyapunov functions for imitative dynamics require a priori specification of the limit state $\mathbf{x}^*$, since those functions basically measure the (Riemannian) geometric distance from $\mathbf{x}^*$. Merikopoulous and Sandholm interpret their Lyapunov function as representing a net gain from motion for a population state, but their gain is mathematically built directly from the aggregate-level dynamic $\mathbf{V}$, not from a revision protocol. It is left how their gain could relate with individual agents' decisions and thus could be interpreted economically.